AS.110.413 Introduction to Topology

Contents

1 Fundamental Concepts

1.1 Section 1: Foundational Concepts

Set Relations and Operations Two set A and B are equal, denoted by $A = B$, if A and B consist of precisely the same objects. A is a ***subset*** of B if every element of A is an element of B (under this definition $A \subset B$ if $A = B$), and we write $A \subset B$. If $A \neq B$ and $A \subset B$, we say A is a ***proper set*** of B and write $A \subsetneq B$.

Given two sets A and B:

(a) The ***union*** of A and B is defined by $A \cup B = \{x \,|\, x \in A \text{ or } x \in B\}$.

(b) The ***intersection*** of A and B is defined by $A \cap B = \{x \,|\, x \in A \text{ and } x \in B\}$.

(c) The ***difference*** of A and B is defined by $A - B = \{x \,|\, x \in A \text{ and } x \notin B\}$.

The arbitrary unions and intersections are defined similarly: they are the union or intersection of all sets in a collection of sets $\mathcal{A}$, and the notations are $\bigcap_{A \in \mathcal{A}} A$ and $\bigcup_{A \in \mathcal{A}} A$. ***DeMorgan's Law***: Suppose U is the universal set, and $\mathcal{A}$ is a collection of sets. Then

$$U - \bigcap_{A \in \mathcal{A}} A = \bigcup_{A \in \mathcal{A}} (U - A) \quad \text{and} \quad U - \bigcup_{A \in \mathcal{A}} A = \bigcap_{A \in \mathcal{A}} (U - A).$$

The ***power set***, denoted by $\mathcal{P}(A)$, is the collection of all subsets of a set A.

Introduction to Logic The statement "if P, then Q" is denoted by $P \Rightarrow Q$.

(a) The ***contrapositive*** of $P \Rightarrow Q$ is the statement "if Q is false, then P is false", namely (not Q) $\Rightarrow$ (not P). The statement and its contrapositive are logically equivalent.

(b) The ***converse*** of $P \Rightarrow Q$ is "if Q, then P", namely $Q \Rightarrow P$.

(c) The ***negation*** of P is the statement not P.

Note that $P \Rightarrow Q$ is logically equivalent to (not P) or Q.

Cartesian Product Given set A and B, the ***Cartesian Product***, denoted by $A \times B$, is defined by $A \times B = \{(a, b) \,|\, a \in A \text{ and } b \in B\}$.

1.2 Section 2: Functions

Rule of Assignment A ***rule of assignment*** is a subset $r \subset C \times D$, where each element of C appears as the first coordinate of at most one ordered pair in r; that is, r is a rule of assignment if $[(c, d) \in r$ and $(c, d') \in r] \Rightarrow [d = d']$.

Given a rule of assignment r, the ***domain*** of r is defined to be the subset of C consisting all first coordinates of elements of r, namely Domain $r = \{c \,|\, \exists d \in D : (c, d) \in r\}$. The ***image set*** is of r is defined to be the subset of C consisting all second coordinates of elements of r, namely Image $r = \{d \,|\, \exists c \in C : (c, d) \in r\}$.

Functions A ***function*** f is a rule of assignment, together with a set B that contains the image set of r. The set B is called the range of f. If a function having domain A and range B, we denote the function as $f : A \to B$.

If $f : A \to B$ and $A_0 \subset A$, we define the ***restriction*** of f to A_0 to be the function mapping A_0 into B whose rule is $\{(a, f(a)) \,|\, a \in A_0\}$.

Injective, Surjective, and Bijective A function $f : A \to B$ is said to be

- ***injective*** (or *one-to-one*) if for each pair of distinct points of A, their images under f are distinct, namely $[f(a) = f(a')] \Rightarrow [a = a']$.
- **surjective** (or *onto*) if every element of B is the image of some element of A under f, namely $[b \in B] \Rightarrow [b = f(a)$ for some $a \in A]$.
- **bijective** if f is both injective and surjective. If f is bijective, there exists an ***inverse*** $f^{-1} : B \to A$ where $f^{-1}(f(a)) = a$ for all $a \in A$ and $f(f^{-1}(b)) = b$ for all $b \in B$.

If $A_0 \subset A$, we denote by $f(A_0)$ the image of A_0 under f, which is set of all images of points of A_0 under f. If $B_0 \subset B$, we denote $f^{-1}(B_0)$ the preimage of B_0 (note that the inverse function does not necessarily exist for preimage to exist).

1.3 Section 3: Relations

Relations and Equivalence Relations A ***relation*** on a set A is a subset C of the cartesian product $A \times A$. We denote xCy if $(x, y) \in C$. An ***equivalence relation*** on a set A is a relation C with the following properties:

(a) Reflectivity: xCx for all $x \in A$.

(b) Symmetry: If xCy, then yCx.

(c) Transitivity: If xCy and yCz, then xCz.

Given an equivalence relation $\sim$ on A, the ***equivalence class*** of x is $E = \{y \,|\, y \sim x\}$. Note that the equivalence classes E and E' are either disjoint or equal.

Order Relations An ***order relations*** on A is a relation C with the following properties:

(a) Comparability: For every $x, y \in A$ such that $x \neq y$, either xCy or yCx.

(b) Nonreflexitivity: For no $x \in A$ does the relation xCx holds.

(c) Transitivity: If xCy and yCz, then xCz.

The symbol $<$ is commonly used to denote an order relation. Suppose $a, b \in A$ and $a < b$, we use (a, b) to denote the set $\{x \,|\, a < x < b\}$. If this set is empty, a is the ***immediate predecessor*** of b, and b is the immediate successor of a.

Two sets A and B (with order relations $<_A$ and $<_B$) have the same ***order type*** if there is a bijective correspondence between them that preserves order; that is, there exists a bijective function $f : A \to B$ such that $a_1 <_A a_2 \Rightarrow f(a_1) <_B f(a_2)$.

Supremum and Infimum $A_0 \subset A$ is bounded above if there is an element $a \in A$ such that $x \leq a$ for all $x \in A_0$; a is called an upper bound of A_0.

> **Supremum, Infimum** If the set of all upper bounds for A_0 has a least element x, then x is called the ***supremum*** (the least upper bound) of A_0 and it is denoted by $x = \sup A_0$. The ***infimum*** is an analogous.

> **Least Upper Bound Property, Greatest Lower Bound Property** An ordered set A is said to have the ***least upper bound property*** if every nonempty subset $A_0 \subset A$ that is bounded above has a least upper bound. The ***greatest lower bound property*** is an analogous.

1.4 Section 4: The Integers and the Real Numbers

Real Numbers Assume there exists a set $\mathbb{R}$ (set of real numbers), two binary operations $+$ and $\cdot$ (addition and multiplication), and an order relation $<$ on $\mathbb{R}$, with the following properties:

- Algebraic properties: (1) associativity, (2) commutativity, (3) identity element, and (4) inverse exists for $(\mathbb{R}, +)$ and $(\mathbb{R}^*, \cdot)$; (5) the distributive law holds; and (6) trichotomy holds for the order relation.
- Order properties: (7) the order relation $<$ has the least upper bound property, and (8) if $x < y$, there exists z such that $x < z < y$.

Integers The set $\mathbb{Z}_+$ of ***positive integers*** is defined by the equation $\mathbb{Z}_+ = \bigcap_{A \in \mathcal{A}} A$ where $\mathcal{A}$ is the collective of all inductive subsets of $\mathbb{R}$ (the subset of $\mathbb{R}$ that contains 1, and $x \in A \Rightarrow x + 1 \in A$ holds). Note that the set $\mathbb{Z}_+$ has no upper bound in $\mathbb{R}$, this is known as the *Archimedean ordering property* holds. The set $\mathbb{Z}$ of ***integers*** is defined to be the set consisiting of the positive integers $\mathbb{Z}_+$, the number 0, and the negative of elements of $\mathbb{Z}_+$.

We have the following theorems:

Theorem 4.1 (Well-ordering property) Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

Theorem 4.2 (Strong induction principle) Let A be a set of positive integers. Suppose that for each positive integer n, the statement $S_n \subset A$ implies the statement $n \in A$. Then $A = \mathbb{Z}_+$.

1.5 Section 5: Cartesian Products

m-Tuple, $\boldsymbol{\omega}$-Tuple Given a set X. Let $m \in \mathbb{Z}_+$, we define an m***-tuple*** of elements of X to be a function $x : \{1, 2, \cdots, m\} \to X$. We define an ω**-tuple** of elements of X to be a function $x : \mathbb{Z}_+ \to X$, and it is called a (infinite) sequence.

We usually denote the function x by $(x_1, \cdots, x_m)$ and its i-th coordinate of x by x_i.

Let $\{A_1, A_2, \cdots\}$ be a family of sets. The Cartesian product of this indexed family $\prod_{i \in \mathbb{Z}_+} A_i = A_1 \times A_2 \times \cdots$ is the set of all ω-tuple of elements of X.

2 Set Theory

2.1 Section 6: Finite Sets

Finite Set, Cardinality A set A is ***finite*** if there is a bijection $f : A \to \{1, \cdots, n\}$ for some positive n. If A is empty, the cardinality is 0, otherwise the cardinality of A is n.

Lemma 6.1 Let A be a set, n be a positive integer, and $a_0 \in A$. There exists a bijection $f : A \to \{1, \cdots, n+1\}$ if and only if there exists a bijection $g : A - \{a_0\} \to \{1, \cdots, n\}$.

Remark: This lemma is saying that removing an element from a finite set will reduce the cardinality by one.

Theorem 6.2

Let A be a set with cardinality n. Let B be a set such that $B \subsetneq A$, then there exists no bijection $g : B \to \{1, \cdots, n\}$ but (if $B \neq \varnothing$) exists a bijection $g : B \to \{1, \cdots, m\}$ for some $m < n$.

Remark: In other words, any proper subset A_0 of a finite set A is finite, and A_0 has a smaller cardinality than A.

Proof: We proceed by induction on n. If $n = 1$, then $B = \varnothing$, so the statement holds. For $n \geq 1$, assume the theorem holds for n. Suppose $f : A \to \{1, \cdots, n+1\}$ is a bijection and $B \subsetneq A$. Choose $x \in B$, there exists a bijection $g : A - \{x\} \to \{1, \cdots, n\}$ (6.1). Suppose $B - \{x\} = \varnothing$, there is a bijection with the set $\{1\}$. Otherwise, suppose $B - \{x\} \neq \varnothing$. Note that $B - \{x\} \subsetneq A - \{x\}$, the inductive hypothesis implies that the desired statements hold for $B - \{x\}$, thus they hold for B (6.1).

The following statements are the corollary of Theorem 6.2:

Corollary 6.4 : $\mathbb{Z}_+$ is not finite.

Corollary 6.5 : The cardinality of a finite set A is uniquely determined by A.

Corollary 6.7

Let B be a nonempty set, then the following are equivalent:

(a) B is finite.

(b) There is a surjective function from a section of the positive integers onto B.

(c) There is an injective function from B into a section of the positive integers.

Proof: $(a) \Rightarrow (b)$: follows immediately from the definition of finite sets.

$(b) \Rightarrow (c)$: Assume there is a surjective function $f : \{1, \cdots, n\} \to B$. Define $g : B \to \{1, \cdots, n\}$ by $g(b) = \min\{x \mid x \in f^{-1}(\{b\})\}$, note that g is uniquely defined by the well-ordering property. g is injective because $f^{(-1)}(\{x\})$ and $f^{(-1)}(\{x'\})$ are disjoint if $x \neq x'$.

$(c) \Rightarrow (a)$: Assume $g : B \to \{1, \cdots, n\}$ is injective, then changing the range gives a bijection into a subset of $\{1, \cdots, n\}$, which is finite.

2.2 Section 8: Countable and Uncountable Sets

Countable, Uncountable A set A is ***countably infinite*** if there is a bijective correspondence $f : A \to \mathbb{Z}_+$. A set is said to be ***countable*** if it is either finite or countably infinite, otherwise it is said to be ***uncountable***.

Theorem 7.1

Let B be a nonempty set, then the following are equivalent:

(a) B is countable.

(b) There is a surjective function $f : \mathbb{Z}_+ \to B$.

(c) There is an injective function $g : B \to \mathbb{Z}_+$.

The proof of Theorem 7.1 uses the following fact:

Lemma 7.2 If C is an infinite subset of $\mathbb{Z}_+$, the C is countably infinite.

Proof: Construct $f : \mathbb{Z}_+ \to C$. Define $f(1)$ to be the least element of C and define $f(n)$ to be the least element of $C - f(\{1, \cdots, n-1\})$, f is well-defined by the well-ordering property and the fact that $C - f(\{1, \cdots, n-1\})$ is nonempty for all n.

We now want to show f is bijective. Given $m < n$, $f(m)$ belongs to $f(\{1, \cdots, n-1\}$ but $f(n)$ does not, so $f(m) \neq f(n)$, thus f is injective. Let $c \in C$, and suppose $c \notin f(\mathbb{Z}_+)$. There exists x such that $f(x) > c$ because C is infinite and f is injective, but $f(x) \leq c$ by the definition of f, resulting in contradiction. Therefore, $c \in f(\mathbb{Z}_+)$, so f is surjective thus bijective.

Corollary 7.3

A subset of a countable set is countable.

Theorem 7.5

A countable union of countable sets is countable.

Theorem 7.6

A finite product of countable sets is countable.

Theorem 7.7 Let X denote the set $\{0, 1\}$, the set X^ω is uncountable.

Proof: Suppose $f : \mathbb{Z}_+ \to X^\omega$, denote $f(n) = (x_{n1}, x_{n2}, \cdots)$. Define $y = (y_1, y_2, \cdots) \in X^\omega$ by $y_n = 0$ if

$x_{nn} = 1$, and $y_n = 1$ if $x_{nn} = 0$. Since $y_n \neq x_{nn}$, $y \neq x_n$ for all $n \in \mathbb{Z}_+$; that is y_n is not in the image set of f, thus f is not surjective. Hence X^ω is uncountable by Theorem 7.1.

We also arrive a more general form of the preceding theorem:

Theorem 7.8 Let A be a set. There is no injective map $f : \mathcal{P} \to A$, and there is no surjective map $g : A \to \mathcal{A}$.

2.3 Section 9: Infinite Sets and the Axiom of Choice

Theorem 9.1

Let A be a set. The following statement about A are equivalent:

(a) There exists a injective function $f : \mathbb{Z}_+ \to A$.

(b) There exists a bijection of A with a proper subset of itself.

(c) A is infinite.

Axiom of Choice

Given a collection $\mathcal{A}$ of disjoint nonempty sets, there exists a set C consisting of exactly one element from each element of $\mathcal{A}$.

Theorem 9.2 (The Existence of a Choice Function) Given a collection $\mathcal{B}$ of nonempty sets (not necessarily disjoint), there exists a function $c : \mathcal{B} \to \bigcup_{B\in\mathcal{B}} B$ such that $c(B)$ is an element of B for each $B \in \mathcal{B}$.

The function c is called a ***choice function*** for the collection $\mathcal{B}$.

2.4 Section 10: Well-Ordered Sets

Well-Ordered Sets A set A with an order relation $<$ is said to be ***well-ordered*** if every nonempty subset of A has a smallest element.

The ways to construct a well-ordered set include:

(a) If A is a well-ordered set, then any subset of A is well-ordered in the restricted order relation.

(b) If A and B are well-ordered sets, then $A \times B$ is well-ordered in dictionary order.

Theorem 10.1

Every nonempty finite ordered set has the order type of a section $\{1, \cdots, n\}$ of $\mathbb{Z}_+$, so it is well-ordered.

Sketch of Proof: First we show that every finite ordered set has a largest element (by induction on the cardinality). Second, we show there is an order-preserving bijection of $\{1, \cdots, n\}$ with A (by induction on n). In the inductive step, let a be the largest element, there is a bijection $f' : \{1, \cdots, n-1\} \to A - \{a\}$ by inductive hypothesis, then construction $f : \{1, \cdots, n\} \to A$ by $f(n) = a$ and $f(x) = f'(x)$ if $x \neq n$, and we can show that f is order-preserving.

Theorem (Well-Ordering Theorem)

If A is a set, there exists an order relation on A that is well-ordering.

Corollary There exists an uncountable well-ordered set.

Section Let X be a well-ordered set. Given $\alpha \in X$, let S_α denote the set $S_\alpha = \{x \mid x \in X \text{ and } x < \alpha\}$. It is called the ***section*** of X by α.

Lemma 10.2

There exists a well-ordered set A having a largest element Ω such that the section S_Ω of A by Ω is uncountable but every other section of A is countable.

Proof: Let B be an uncountable well-ordered set, and denote the least element of B by b. Define $C := \{0,1\} \times B$ with dictionary order, note that C is also well-ordered. $S_{(1,x)}$ is uncountable for all $x \in B$ and $S_{(0,b)} = \varnothing$ is countable. Let Ω be the smallest element of C for which S_Ω is uncountable, then $A = S_\Omega \cup \{\Omega\}$ has the desired property.

Remark: We called S_Ω the minimal countable well-ordered set.

Theorem 10.3 If A is a countable subset of S_Ω, then A has an upper bound in S_Ω.

3 Topological Spaces and Specific Topologies

3.1 Section 12: Topological Spaces

Topology, Topological Spaces A ***topology*** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. The empty set $\varnothing$ and the set X are in $\mathcal{T}$.
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a ***topological space***.

Remark: The condition (2) and (3) is saying that topology is closed under unions and finite intersections.

Suppose X is nonempty. $\mathcal{T} = \{\varnothing, X\}$ is a topology on X, it is said to be the *indiscrete topology* (*trivial topology*), and it is the minimal topology on X. $\mathcal{T} = \mathcal{P}(X)$ is also a topology on X, it is said to be the *discrete topology*, and it is the maximal topology on X.

Open Set Suppose $\mathcal{T}$ is a topology on X. A subset U of X is an ***open set*** of X if U belongs to $\mathcal{T}$.

Finer, Coarser Suppose $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T} \subset \mathcal{T}'$, $\mathcal{T}$ is said to be ***coarser*** than $\mathcal{T}'$, and $\mathcal{T}'$ is said to be ***finer*** than $\mathcal{T}$. If $\mathcal{T} \subsetneq \mathcal{T}'$, $\mathcal{T}$ is said to be *strictly coarser* than $\mathcal{T}'$, and $\mathcal{T}'$ is said to be *strictly finer* than $\mathcal{T}$. We say $\mathcal{T}$ is ***comparable*** with $\mathcal{T}'$ if either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{T}' \subset \mathcal{T}$.

3.2 Section 13: Basis for Topology

3.2.1 Basis for Topology

Basis, Open Subset If X is a set, a ***basis*** for a topology on X is a collection $\mathcal{B}$ of subsets of X (Called ***basis elements***) such that:

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to the intersection of two basis element B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, the we define the topology ***$\mathcal{T}$ generated by $\mathcal{B}$*** as follows: A subset U of X is said to be ***open*** in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Remark: The collection $\mathcal{T}$ generated by the basis $\mathcal{B}$ is a topology on X.

Proof: (1) It is not hard to show that $\varnothing, X \in \mathcal{T}$ by definition.

(2) For all $x \in \bigcup_\alpha U_\alpha$ where $U_\alpha \in \mathcal{T}$, $x \in U_\beta$ for some β. Since U_β is open in X, there is $B \in \mathcal{B}$ such that $x \in B \subset U_\beta$, thus $x \in B \subset \bigcup_\alpha U_\alpha$. Then $\bigcup_\alpha U_\alpha$ is open in X by definition.

(3) For all $x \in \bigcap_\alpha U_\alpha$, there exists $B_\alpha \in \mathcal{B}$ for each α, such that $x \in B_\alpha \subset U_\alpha$, because U_α is open in $\mathcal{T}$ for all α. There exists $B \in \mathcal{B}$ such that $B \subset \bigcap_\alpha B_\alpha$ by the definition of basis. Then $x \in B \subset \bigcap_\alpha B_\alpha \subset \bigcap_\alpha U_\alpha$. Therefore, $\bigcup_\alpha U_\alpha$ is open in X.

Lemma 13.1

Let X be a set, let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Remark: Equivalently, U is a open set in $\mathcal{T}_\mathcal{B}$ if and only if U is an union of sets in $\mathcal{B}$.

Proof: ($\Leftarrow$) Given $\mathcal{B}$, each $B \in \mathcal{B}$ is an element of $\mathcal{T}$. Since $\mathcal{T}$ is a topology, union of $\mathcal{B}$ is in $\mathcal{T}$. ($\Rightarrow$) Given $U \in \mathcal{T}$, for all $x \in U$, choose $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x \in U} B_x$, so U is a union of elements of $\mathcal{B}$. This completes the proof.

Lemma 13.2

Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element $C \in \mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Lemma 13.3

Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively, on X. Then the following are equivalent:

(a) $\mathcal{T}'$ is finer than $\mathcal{T}$.

(b) For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof: ($\Leftarrow$) Given $U \in \mathcal{T}$. For all $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$, and there is $B' \in \mathcal{B}'$ such that $B' \subset B$ by condition (b). Then $x \in B' \subset B \subset U$, so $U \in \mathcal{T}'$.

($\Rightarrow$) Suppose $\mathcal{T} \subset \mathcal{T}'$. For all $B \in \mathcal{B} \subset \mathcal{T}$, we have $B \in \mathcal{T}'$. Then for all $x \in B$, there is $B' \in \mathcal{B}'$ such that $x \in B' \subset B$ by the definition of open sets.

Subbasis A ***subbasis*** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The ***topology generated by the subbasis*** $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of S..

Remark: The topology generated by the subbasis $\bigcup \mathcal{T}_\alpha$ is the smallest topology containing all the collections $\mathcal{T}_\alpha$.

In order to show that $\mathcal{T}$ is a topology, it is suffice to show that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis (Lemma 13.1). Condition (1): Since the union of $\mathcal{S}$ equals X, for all $x \in X$, x belongs to an element of $\mathcal{S}$ thus to an element of $\mathcal{B}$. Condition (2): For all $B_1, B_2 \in \mathcal{B}$, B_1 and B_2 are finite intersections of $\mathcal{S}$, so $B_1 \cap B_2$ is also a finite intersection of $\mathcal{S}$, thus $B_1 \cap B_2 \in \mathcal{B}$.

3.2.2 Standard Topology

Standard Topology, Lower Limit Topology ($\mathbb{R}_l$), K-Topology ($\mathbb{R}_K$)

- If $\mathcal{B}$ is the collection of all open intervals in the real line, $(a, b) = \{x \mid a < x < b\}$, the topology generated by $\mathcal{B}$ is called the ***standard topology*** on the real line.
- If $\mathcal{B}'$ is the collection of all half-open intervals of the form $[a, b) = \{x \mid a \le x < b\}$, where $a < b$, the topology generated by $\mathcal{B}'$ is called the ***lower limit topology*** on $\mathbb{R}$, and we denote it by $\mathbb{R}_l$.
- Let K denote the set of all numbers of the form $1/n$ for $n \in \mathbb{Z}_+$, and let $\mathcal{B}''$ be the collection of all open intervals (a, b) along with all set of the form $(a, b) - K$. The topology generated by $\mathcal{B}''$ is called the ***K-topology*** on $\mathbb{R}$, and we denote it by $\mathbb{R}_k$.

Lemma 13.4 The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

Sketch: Clearly $\mathbb{R}_l$ and $\mathbb{R}_K$ are finer than the standard topology. Note that $[-1, 0)$ is a basis element of $\mathbb{R}_l$, $(-1, 1) - K$ is a basis element of $\mathbb{R}_K$, but they are not basis elements of one another nor the standard topology. This completes the proof.

3.3 Section 14: The Order Topology

Order Topology Let X be a set with a simple order relation; assume X has more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.

(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

The collection $\mathcal{B}$ is a basis for a topology on X, which is called the ***order topology***.

If X is an ordered set, and a is an element of X, the following subsets are called ***rays***: (1) $(a, +\infty) = \{x \mid x > a\}$, (2) $(-\infty, a) = \{x \mid x < a\}$, (3) $[a, +\infty) = \{x \mid x \geq a\}$, and (4) $(-\infty, a] = \{x \mid x \leq a\}$. The first two types are open rays, and the last two types are called closed rays.

Every basis element for the order topology equals a finite intersection of open rays. Then the open rays form a subbasis for the order topology of X.

3.4 Section 15: The Product Topology on $\mathbf{X} \times \mathbf{Y}$

Product Topology Let X and Y be topological spaces. The ***product topology*** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$ where U is an open subset of X and V is an open subset of Y.

The product topology is well-define, because the collection $\mathcal{B}$ is a basis:

(1) For all $(x, y) \in X \times Y$, note that $X \times Y$ is itself a basis element, so $(x, y) \in X \times Y \subset \mathcal{B}$.

(2) Suppose $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$. Note that $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \times V_2)$ and the latter set is a basis element because $U_1 \cap U_2$ and $V_1 \times V_2$ are open in X and Y, respectively. Then there exists $B = (U_1 \cap U_2) \times (V_1 \times V_2) \in \mathcal{B}$ such that $x \in B \subset (U_1 \times V_1) \cap (U_2 \times V_2)$.

Theorem 15.1

If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection $\mathcal{D} = \{B \times C \,|\, B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$.

Proof: We apply Lemma 13.2. Given an open set W and a point $(x, y) \in W$, there exists a basis element $U \times V$ such that $(x, y) \in U \times V \subset W$ by the definition of open set. Since U and V are open in X and Y, respectively, there exists $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B \subset U$ and $y \in C \subset V$. Then $B \times C$ is an element in $\mathcal{D}$ for which $(x, y) \subset B \times C \subset W$. Hence by Lemma 13.2, $\mathcal{D}$ is a basis for $X \times Y$.

Projection Let $\pi_1 : X \times Y \to X$ be defined by $\pi_1(x, y) = x$; let $\pi_2 : X \times Y \to Y$ be defined by $\pi_2(x, y) = y$. The maps π_1 and π_2 are called ***projections*** of $X \times Y$ onto its first and second factors, respectively.

Theorem 15.2

The collection $\mathcal{S} = \{\pi_1^{-1}(U) \,|\, U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) \,|\, V \text{ is open in } Y\}$ is a subbasis for the product topology on $X \times Y$.

Proof: Let $\mathcal{T}$ denotes the product topology and $\mathcal{T}'$ denotes the topology generated by $\mathcal{S}$. Since every element of $\mathcal{S}$ belongs to $\mathcal{T}$, so do arbitrary unions of finite intersections of elements of $\mathcal{S}$, thus $\mathcal{T}' \subset \mathcal{T}$. Every basis element $U \times V$ of $\mathcal{T}$ can be written as $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$, the finite intersection of elements of $\mathcal{S}$, so $U \times V \in \mathcal{T}'$, thus $\mathcal{T} \subset \mathcal{T}'$.

3.5 Section 16: The Subspace Topology

Subspace Topology Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, then collection

$$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$$

is a topology on Y, called the ***subspace topology***. With this topology, Y is called a subspace of X, its open sets consist of all intersections of open sets of X with Y.

The collection $\mathcal{T}_Y$ is a topology by definition,

(1) $\varnothing$ and Y is in $\mathcal{T}_Y$: $\varnothing = Y \cap \varnothing$ and $Y = Y \cap X$, so $\varnothing, Y \in \mathcal{T}_Y$

(2) $\mathcal{T}_Y$ is closed under arbitrary unions: $\bigcup_\alpha U_\alpha$ is open in X, then $\bigcup_\alpha (Y \cap U_\alpha) = Y \cap (\bigcup_\alpha U_\alpha) \in \mathcal{T}_Y$,

(3) $\mathcal{T}_Y$ is closed under finite intersections: $\bigcap_{i=1}^n U_i$ is open in X, then $\bigcap_{i=1}^n (Y \cap U_i) = Y \cap (\bigcap_{i=1}^n U_i) \in \mathcal{T}_Y$.

Lemma 16.1

If $\mathcal{B}$ is a basis for the topology of X, then the collection

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof: We apply Lemma 13.2. Given an open set U and $y \in U \cap Y$, we can choose $B \in \mathcal{B}$ such that $y \in B \subset U$. Then $y \in B \cap Y \subset U \cap Y$. The desired result follows from Lemma 13.2.

Lemma 16.2

Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof: U is open in Y, so $U = Y \cap V$ for some set V open in X. Then $Y \cap V$ is open because it is the finite intersection of open sets.

The following theorems explore the relation between subspace topology and the order and product topologies.

Theorem 16.3

If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as the subspace of $X \times Y$.

Proof: The sets $(U \times V) \cap (A \times B)$ and $(U \cap A) \times (V \cap B)$ are the general basis elements for the subspace topology on $A \times B$ (denoted by $\mathcal{T}_1$) and product topology on $A \times B$ (denoted by $\mathcal{T}_2$), respectively. Suppose $x \in (U \cap A) \times (V \cap B)$, where $U \times V$ is open in $X \times Y$. Then $x \in (U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$, so $\mathcal{T}_1$ is finer than $\mathcal{T}_\in$ (lemma 13.3). Similarly, $\mathcal{T}_2$ is finer than $\mathcal{T}_1$. Hence the two topologies are equal.

However, the preceding statement does not holds for order topology. The order topology on Y is not necessarily the same as the topology that Y inherits as a subspace of X.

Convex Given an ordered set X, we say that a subset Y of X is ***convex*** in X if for each pair of points $a < b$ of Y, the entire interval (a, b) of points of X lies in Y. Note that intervals and rays in X are convex in X.

Theorem 16.4

Let X be an ordered set in the order topology; let Y be a subset of X that is convex in X. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

Proof: Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ denotes the order topology and subspace topology, respectively.

($\mathcal{T}_2 \subset \mathcal{T}_1$) Recall that Y is convex in X. The intersection of Y with the ray $(a, +\infty)$ or $(-\infty, a)$ in X is either an open ray of Y, Y it self, or empty. Since the set $(a, +\infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis for the $\mathcal{T}_2$, and each is open in the $\mathcal{T}_1$, then $\mathcal{T}_2 \subset \mathcal{T}_1$.

($\mathcal{T}_1 \subset \mathcal{T}_2$) Note that any open ray of Y equals the intersection of an open ray of X with Y, so it is open in $\mathcal{T}_2$. Since open rays of Y are a subbasis of $\mathcal{T}_1$, $\mathcal{T}_1 \subset \mathcal{T}_2$.

4 Properties of Topologies and Continuous Functions

4.1 Section 17: Closed Sets and Limit Points

4.1.1 Closed Sets

Closed Set A subset A of a topological space X is said to be ***closed*** if the set $X - A$ is open.

Remark: A set can be both open and closed (e.g., open sets in the discrete topology of X), or neither open nor closed (e.g., $(a, b]$ in $\mathbb{R}$).

Suppose Y is a subspace of X, a set A is closed in Y if A is a subset of Y and A is closed in the subspace topology (that is, $Y - A$ is open in Y).

The following theorems are the analogous of definition of topology, the definition of subspace topology, and Lemma 16.2, using closed sets.

Theorem 17.1

Let X be a topological space. Then the following conditions hold:

(1) $\varnothing$ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

Proof: (1) $\varnothing$ and X are the complements of each other, so they are closed.

(2) Suppose $\{A_\alpha\}$ is a collection of closed sets, then $X - \bigcap_\alpha A_\alpha = \bigcup_\alpha (X - A_\alpha)$ by DeMorgan's Law. Since $X - A_\alpha$ is open for all α, so do the arbitrary unions, thus $\bigcap_\alpha A_\alpha$ is closed.

(3) Suppose A_i is closed for $i = 1, \cdots, n$, then $X - \bigcup_{i=1}^n A_\alpha = \bigcap_{i=1}^n (X - A_\alpha)$ by DeMorgan's Law. Since $X - A_i$ is open for all i, so do the finite intersections, thus $\bigcup_{i=1}^n A_\alpha$ is closed.

Theorem 17.2

Let Y be a subspace of X. Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

Proof: ($\Leftarrow$) Suppose $A = C \cap Y$, where C is closed in Y. Taking the relative complement of both sides yields $Y - A = Y - (C \cap Y) = Y - C = Y \cap (X - C)$. Note that Y and $X - C$ are open, so $Y - A$ is open, thus A is closed.

($\Rightarrow$) Suppose A is closed in Y. The set $Y - A$ is open in Y, so $Y - A = Y \cap U$ for some U open in X. Taking the relative complement of both sides yields $A = Y - (U \cap Y) = Y \cap (X - U)$. Since U is open, $X - U$ is closed, which completes the proof.

Theorem 17.3 Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Proof: There exists a set C closed in X such that $A = Y \cap C$ (Theorem 17.2). Since Y and C are closed in X, $A = Y \cap C$ is closed in X (Theorem 17.1(b)).

4.1.2 Closure and Interior of a Set

Interior, Closure Given a subset A of a topological space X. The ***interior*** of A, denoted by Int A, is defined as the union of all open sets contained in A. The ***closure*** of A, denoted by $\bar{A}$, is defined as the intersection of all closed sets containing A.

Furthermore, Int $A \subset A \subset \bar{A}$. If A is open, $A =$ Int A, and if A is closed, $A = \bar{A}$.

Theorem 17.4

Let Y be a subspace of X, let A be a subset of Y, let $\overline{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

Proof: Let $\bar{A}_Y$ denote the closure of A in Y. Since $\bar{A}$ is closed in Y, $\bar{A} \cap Y$ is closed in Y (Theorem 17.2), so $\bar{A}_Y \subset \bar{A} \cap Y$. On the other hand, note that $\bar{A}_Y$ is closed in Y, so $\bar{A}_Y = C \cap Y$ for some C closed in X containing $\bar{A}_Y$. Then $\bar{A} \cap Y \subset \bar{A}_Y$, since $\bar{A} \subset C$.

Theorem 17.5

Let A be a subset of the topological space X.

(a) Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.

(b) Supposing the topology of X is given by a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A.

Proof: (a) ($\Rightarrow$) Suppose there exists U containing x such that $U \cap A = \varnothing$. The complement $X - U$ is closed, so $\bar{A} \subset X - U$. Note that $x \notin X - U$, $x \notin \bar{A}$. ($\Leftarrow$) Conversely, suppose $x \notin \bar{A}$. The set $X - \bar{A}$ is open and $x \in X - \bar{A}$, but $(X - \bar{A}) \cap A = \varnothing$, so there exists open set U containing x that does not intersect A.

(b) The statement follows immediately from the fact that every basis element is open in X and every open set is the union of basis elements.

Note: We called that U is a ***neighborhood*** of x if U is an open set containing x.

4.1.3 Limit Points

Using the concept of limit point, we obtain another way of describing the closure of a set.

Limit Point If A is a subset of the topological space X and if x is a point of X, we say that x is a ***limit point*** of A if every neighborhood of x intersects A in some points other than x it self. In other words, x is a limit point of A if it belongs to the closure of $A - \{x\}$.

That is, x is a limit point if $(U \cap A) - \{x\} \neq \varnothing$ for all open set U containing x.

Theorem 17.6

Let A be a subset of the topological space X, let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

Proof: Note that $A \subset \bar{A}$, and $A' \subset \bar{A}$ by the definition of A'. Then $A' \cap A \subset \bar{A}$. Conversely, suppose $x \in \bar{A}$. If $x \notin A$, then for all open set U containing x, $(U \cap A) - \{x\} = U \cap A \neq \varnothing$ (Theorem 17.5), so $x \in A'$. Then $\bar{A} \subset A' \cap A$.

Corollary 17.7

A subset of a topological space is closed if and only if it contains all its limit points.

4.1.4 Hausdorff Spaces

Hausdorff Space A topological space X is called a ***Hausdorff space*** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.

Theorem 17.8 Every finite point set in a Hausdorff space X is closed.

Proof: It is suffice to prove every $\{x_0\}$ is closed. For all $x \in X$ such that $x \neq x_0$, x has a neighborhood U such that $x_0 \notin U$. Then $x \notin \overline{\{x_0\}}$ by definition, so $\{x_0\} = \overline{\{x_0\}}$, thus it is closed.

Alternative Proof: Given a point $x_0 \in A$, for all $x \notin A$, there is an open set U_x containing x such that $x_0 \notin U_x$. Put $K_{x_0} = \bigcup_{x \notin A} U_x$, then K_{x_0} is an open set such that $x_0 \notin K_{x_0}$ and $X - A \subset K_{x_0}$. Put $K = \bigcap_{x_0 \in A} K_{x_0}$ is an open set such that $X - A \subset K$ and $A \cap K = \varnothing$, that is, $K = X - A$ is an open set. Hence A is closed.

Remark: The condition in this theorem is called the ***T_1 axoim***, and it is indeed weaker than the Hausdorff condition.

Theorem 17.9 Let X be a space satisfying T_1 axoim (every finite point set is closed); let A be a subset of X. Then the point x is a limit point of A if and only if neighborhood of x contains infinitely many points of A.

Theorem 17.10

If X is a Hausdorff space, then a sequence of points of X converges to at most one point of X.

Theorem 17.11

(a) Every simply ordered set is a Hausdorff space in the order topology.

(b) The product of two Hausdorff spaces is a Hausdorff space.

(c) A subspace of a Hausdorff space is a Hausdorff space.

4.2 Section 18: Continuous Functions

4.2.1 Continuity of a Function

Continuity Let X and Y be topological spaces. A function $f : X \to Y$ is said to be ***continuous*** if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.

To prove a function is continuous, it is suffices to show that the inverse image of every basis element or every subbasis element is open.

Theorem 18.1

Let X and Y be topological spaces; let $f : X \to Y$. Then the following are equivalent:

(1) f is continuous.

(2) For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.

(3) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Proof: **(1) ⇒ (2)**: Given $x \in \bar{A}$. For all neighborhoods V of $f(x)$, $f^{-1}(V)$ is a neighborhood of x, thus it intersects A (Theorem 17.5). Then there exists $y \in A$ such that $y \in f^{-1}(V)$, and note that $f(y) \in f(A)$, so V intersects $f(A)$. Therefore, $f(x) \in \overline{f(A)}$.

(2) ⇒ (3): Suppose B is closed in Y, let $A = f^{-1}(B)$. Note that $f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B$. Then for $x \in \bar{A}$, $x \in f^{-1}(B) = A$, so $\bar{A} \subset A$.

(3) ⇒ (1): Given V open in Y. The set $Y - V$ is closed, so $f^{-1}(Y - V) = X - f^{-1}(V)$ is closed by (3), thus $f^{-1}(V)$ is open in X.

(1) ⇒ (4): Suppose V is a neighborhood of $f(x)$, then $U = f^{-1}(V)$ is a desired neighborhood of x.

(4) ⇒ (1): Suppose V is open in Y. For all $x \in X$ such that $f(x) \in V$, by (4), there exists a neighborhood U_x of x such that $f(U_x) \subset V$, namely $U_x \subset f^{-1}(V)$. Note that $f^{-1}(V) = \bigcup U_x$, a union of open sets, so it is open.

4.2.2 Homeomorphisms

Homeomorhpism Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a ***homeomorphism***.

Remark: Homeomorphism preserve all the topological properties of a given space.

Topological Imbedding Let X and Y be topological spaces; let $f : X \to Y$ be an injective map. If the bijection $f' : X \to f(X)$, obtained by restricting the range of f and consider $f(X)$ as a subspace of Y, is a homeomorphism of X with $f(X)$, then the map $f : X \to Y$ is a ***topological imbedding*** of X in Y.

4.2.3 Constructing Continuous Functions

Theorem 18.2 (Rules for constructing continuous functions) Let X, Y, Z be topological spaces.

(a) (Constant function) If $f : X \to Y$ maps all of X into the single point y_0 of Y, then f is continuous.

(b) (Inclusion) If A is a subspace of X, the inclusion function $j : A \to X$ is continuous.

(c) (Composites) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.

(d) (Restricting the domain) If $f : X \to Y$ is continuous, and if A is a subspace of X, the the restricted function $f|_A : A \to Y$ is continuous.

(e) (Restricting or expanding the range) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space containing Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.

(f) (Local formulation of continuity) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U_\alpha}$ is continuous for each α.

Theorem 18.3 (The pasting lemma) Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$, and $h(x) = g(x)$ if $x \in B$.

Proof: Let C be a closed subset of Y. Now $h^{-1}(C) = f^{-1}(C) \cap g^{-1}(C)$. Since f, g are continuous, $f^{-1}(C)$ and $g^{-1}(C)$ are closed (Theorem 18.1). Then $h^{-1}(C)$ is closed, thus h is continuous (Theorem 18.1).

Remark: This theorem also holds if A and B are open sets in X, which is a special case of the local formulation of continuity (Theorem 18.2 (f)).

Theorem 18.4 (Maps into products) Let the coordinate function $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$. Then f is continuous if and only if the functions $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous.

Sketch of Proof: ($\Rightarrow$) Suppose f is continuous. Notice that $f_1(a) = \pi_1(f(a))$ and $f_2(a) = \pi_2(f(a))$. Obviously π_1 and π_2 are continuous, so f_1, f_2 are continuous (Theorem 18.1 (c)).

($\Leftarrow$) Conversely, suppose f_1 and f_2 are continuous. For each basis element $U \times V$, $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$, which is open since both $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open. Then f is continuous.

4.3 Section 19: The Product Topology

J-tuple Let J be an index set. Given a set X, we define ***J-tuple*** of elements of X to be a function $\mathbf{x} : J \to X$. We denote the α-th coordinate of $\mathbf{x}$ by x_α, the function $\mathbf{x}$ by $(x_\alpha)_{\alpha \in J}$, and the set of all J-tuple of elements of X by X^J.

Cartesian Product Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $X = \bigcup_{\alpha \in J} A_\alpha$. The **cartesian product** of this indexed family, denoted by $\prod_{\alpha \in J} A_\alpha$, is defined to be the set of all J-tuple $(x_\alpha)_{\alpha \in J}$ of elements of X such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$.

That is, the set of all functions $\mathbf{x} : J \to \bigcup_{\alpha \in J} A_\alpha$ such that $\mathbf{x}(\alpha) \in A_\alpha$ for each $\alpha \in J$.

Box Topology Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. Let us take as a basis for a topology on the product space $\prod_{\alpha \in J} X_\alpha$ the collection of all sets of the form $\prod_{\alpha \in J} U_\alpha$ where U_α is open in X_α, for each $a \in J$. The topology generated by this basis is called the ***box topology***.

Product Topology Let $\mathcal{S}_\beta$ denote the collection $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) \,|\, U_\beta \text{ open in } X_\beta\}$, and let $\mathcal{S}$ denote the union of these collections, $\mathcal{S} = \bigcup_{\beta \in J} \mathcal{S}_\beta$. The topology generated by the subbasis $\mathcal{S}$ is called the ***product topology***, and $\bigcap_{\alpha \in J} X_\alpha$ is called a ***product space***.

Theorem 19.1 (Comparison of the box and product topologies) The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α. The product topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ except for finitely many values of α.

Theorem 19.2

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. The collection of sets of the form $\prod_{\alpha \in J} B_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ for each α, will serve as a basis for the box topology.

The collection of all sets of the same form, where $B_\alpha \in \mathcal{B}_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology $\prod_{\alpha \in J} X_\alpha$.

Theorem 19.3 Let A_α be a subspace of X_α, for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given the box topology, or if both product are given the product topology.

Theorem 19.4 If each space X_α is a Hausdorff space, then $\prod X_\alpha$ is a Hausdorff space in both box and product topologies.

Theorem 19.5 Let $\{X_\alpha\}$ be an index family of spaces; let $A_\alpha \subset X_\alpha$ for each α. If $\prod X_\alpha$ is given either the product or the box topology, then $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$.

Proof: $(\subset)$ Suppose $\boldsymbol{x} = (x_\alpha) \in \prod \overline{A_\alpha}$, and let $U = \prod U_\alpha$ be a basis element in either topologies containing

$\boldsymbol{x}$. Since $x_\alpha \in \overline{A_\alpha}$ and U_α is open in X_α, $U_\alpha \cap A_\alpha \neq \varnothing$. Then $U \cap \prod A_\alpha = \prod(U_\alpha \cap A_\alpha) \neq \varnothing$, so $\boldsymbol{x} \in \overline{\prod A_\alpha}$, and thus $\prod \overline{A_\alpha} \subset \overline{\prod A_\alpha}$.

($\supset$) Conversely, suppose $\boldsymbol{x} = (x_\alpha) \in \overline{\prod A_\alpha}$ in either topologies. For each $\beta \in J$, suppose U_β is an arbitrary open set in X_β containing x_β. Since $\pi_\beta^{-1}(U_\beta)$ is open, there is $\boldsymbol{y} = (y_\alpha)$ such that $\boldsymbol{y} \in \pi^{-1}(U_\beta) \cap \prod A_\alpha$. Then $y \in U_\beta \cap A_\beta$, so the intersection is not empty, and it follows that $x_\beta \in \overline{A_\beta}$.

Theorem 19.6 Let $f : A \to \prod X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Proof: ($\Rightarrow$) Let β be an arbitrary element of J, and U_β is open in X_β. Put $U = \pi_\beta^{-1}(U_\beta)$, which is open in $\prod X_\alpha$. Notice that $f^{-1}(U)$ is open in the product space, and $f^{-1}(U) = (\bigcap f_\alpha^{-1}(X_\alpha)) \cap f_\beta^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$. Then $f_\beta^{-1}(U_\beta)$ is open, thus f_β is continuous.

($\Leftarrow$) Suppose $U = \prod U_\alpha$ is open in $\prod X_\alpha$, U_α is open for all α. Since $f^{-1}(U) = \bigcap f_\alpha^{-1}(U_\alpha)$, and $U_\alpha = X_\alpha$ for all but finitely many α, $f^{-1}(U)$ can be written as the intersection of finite $f_\alpha^{-1}(U_\alpha)$. For every α, $f_\alpha^{-1}(U_\alpha)$ is open since f_α is continuous, so $f^{-1}(U)$ is open, and thus f is continuous.

4.4 Section 20: The Metric Topology

4.4.1 The Metric Topology

Metric A ***metric*** on a set X is a function $d : X \times X \to \mathbb{R}$ having the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Given a metric d on X, the number $d(x, y)$ is called the distance between x and y in the metric d. Given $\varepsilon > 0$, the set $B_d(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\}$ is called the ε-***ball*** centered at x, or $B(x, \varepsilon)$.

Metric Topology If d is a metric on the set X, then the collection of all ε-ball $B_d(x, \varepsilon)$ for $x \in X$ and $\varepsilon > 0$ is a basis for a topology on X, called the ***metric topology*** induced by d.

Remark: A set U is open in the metric topology induced by d if and only if for each $y \in U$, there is a $\delta > 0$ such that $B(y, \delta) \subset U$.

Metric Space If X is a topological space, X is said to be ***metrizable*** if there exists a metric d on X. A metric space is a metrizable space X together with a specific metric d that gives the topology of X.

Boundedness, Diameter Let X be a metric space with metric d. A subset A of X is said to be ***bounded*** if there is some number M such that $d(x, y) \leq M$ for every pair $x, y \in A$. If A is bounded and nonempty, the ***diameter*** of A is defined to be the number $\text{diam } A = \sup\{d(x, y) \mid x, y \in A\}$.

Theorem 20.1

Let X be a metric space with metric d. Define $\bar{d} : X \times X \to \mathbb{R}$ by the equation $\bar{d} = \min\{d(x, y), 1\}$. Then $\bar{d}$ is a metric that induces the same topology as d.

Remark: $\bar{d}$ is called the ***standard bounded metric*** corresponding to d.

Proof: We first need to show $\bar{d}$ is a metric. The first two conditions are trivial. The triangle inequality holds because $\bar{d}(x, z) \leq \min\{d(x, y) + d(y, z), 1\} \leq \min\{d(x, y), 1\} + \min\{d(y, z), 1\} = \bar{d}(x, y) + \bar{d}(y, z)$.

Now we want to prove the two topologies are the same. It is not hard to prove the collection of ε-ball with $\varepsilon < 1$ forms a basis for the metric topology (Lemma 13.2), and the collection of ε-ball with $\varepsilon < 1$ are the same for d and $\bar{d}$. It follows that d and $\bar{d}$ induce the same topology.

Lemma 20.2

Let d and d' be two metrics on the set X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in X$ and each $\varepsilon > 0$, there exists a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \varepsilon)$.

Sketch of Proof: Note that $B_{d'}(x, \delta)$ and $B_d(x, \varepsilon)$ are basis elements of topologies induced by d and d', respectively, so the desired statements follows from Lemma 13.3.

4.4.2 Euclidean Metric and Square Metric

Norm, Euclidean Metric, and Square Metric Given $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we define the ***norm*** of $\boldsymbol{x}$ by the equation $\|\boldsymbol{x}\| = (x_1^2 + \cdots + x_n^2)^{1/2}$. We defined the ***Euclidean metric*** d on $\mathbb{R}^n$ by the equation $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$, and we define the ***square metric*** ρ on $\mathbb{R}^n$ by the equation $\rho(\boldsymbol{x}, \boldsymbol{y}) = \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}$.

Theorem 20.3

The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.

Proof: We will first prove $\mathcal{T}_d = \mathcal{T}_\rho$. Notice that $\rho(\boldsymbol{x}, \boldsymbol{y}) \leq d(\boldsymbol{x}, \boldsymbol{y}) \leq \sqrt{n} \cdot \rho(\boldsymbol{x}, \boldsymbol{y})$. Then $B_d(\boldsymbol{x}, \varepsilon) \subset B_\rho(\boldsymbol{x}, \varepsilon)$, so the topology induced by d and ρ are the same (Lemma 20.2).

Then we will prove $\mathcal{T}_\rho$ is the same as the product topology $\mathcal{T}$. Every basis element of $\mathcal{T}_\rho$ is itself a basis element of the product topology, so $\mathcal{T}_\rho \subset \mathcal{T}$. In addition, suppose $B = \prod_{i=1}^n B_i$ is a basis element of the product topology and $\boldsymbol{x}$ is an arbitrary point in it. For each i, there exists $B(x_i, \varepsilon_i)$ such that $x_i \in B(x_i, \varepsilon_i) \subset B_i$ since B_i is open. Then put $\varepsilon = \min\{\varepsilon_1, \cdots, \varepsilon_n\}$, we see that $B_\rho(\boldsymbol{x}, \varepsilon) \subset B$, thus $\mathcal{T} \subset \mathcal{T}_\rho$ (Lemma 13.3).

4.4.3 The Uniform Topology

Uniform Topology Given an index set J, and given points $\mathbf{x} = (x_\alpha)_{\alpha \in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha \in J}$ on $\mathbb{R}^J$. Let us define a metric $\bar{\rho}$ on $\mathbb{R}^J$ by the equation $\bar{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$, where $\bar{d}$ is the standard bounded metric on $\mathbb{R}$. The metric $\bar{\rho}$ is called the ***uniform metric*** on $\mathbb{R}^J$, and the topology it induces is called the ***uniform topology***.

Theorem 20.4 The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology; these three topologies are all different if J if infinite.

Proof: Suppose $\boldsymbol{x} = (x_\alpha)_{\alpha \in J}$ and a basis element $\prod U_\alpha$ of the product topology are given. Let $i = 1, \cdots, n$ be the indices for which $U_{\alpha_i} \neq \mathbb{R}$, then there exists $B_{\alpha_i} = B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i)$ such that $x_{\alpha_i} \in B_i \subset U_{\alpha_i}$. Put $\varepsilon = \min\{\varepsilon_i\}$, then $\boldsymbol{x} \in B_{\bar{\rho}}(\boldsymbol{x}, \varepsilon) \subset \prod U_\alpha$, thus the uniform topology is finer than the product topology.

On the other hand, let B be the ε-ball centered at $\mathbf{x}$ in the $\bar{\rho}$ metric. Then the box neighborhood $U =$

$\prod(x_\alpha - \frac{1}{2}\varepsilon, x_\alpha + \frac{1}{2}\varepsilon)$ of $\mathbf{x}$ is contained in B, since for $\mathbf{y} \in U$, $\bar{d}(x_\alpha, y_\alpha) < \varepsilon/2$, so $\bar{\rho}(\mathbf{x}, \mathbf{y}) \leq \frac{1}{2}\varepsilon < \varepsilon$.

The proof of all three topologies are different is omitted.

Theorem 20.5 Let $\bar{d}$) be the standard bounded metric on $\mathbb{R}$. If $\mathbf{x}$ and $\mathbf{y}$ are two points of $\mathbb{R}^\omega$, defined $D(\mathbf{x}, \mathbf{y}) = \sup\{\bar{d}(x_i, y_i)/i\}$. Then D is a metric that induces the product topology on $\mathbb{R}^\omega$.

Proof: The proof of D is a metric is omitted. Suppose $U = \prod U_i$ is a basis element of the product topology, where $U_i \neq \mathbb{R}$ for $i = \alpha_1, \cdots, \alpha_n$ and $U_i = \mathbb{R}$ otherwise, and $\mathbf{x} \in U$ is given. For each i, there exists $(x_i - \varepsilon_i, x_i + \varepsilon) \subset U_i$ since U_i is open. Put $\varepsilon = \min\{\varepsilon_i/i\}$, then $\mathbf{x} \in B_D(\mathbf{x}, \varepsilon) \subset U$, so $\mathcal{T}_D$ is finer.

Conversely, suppose $B_D(\mathbf{x}, \varepsilon)$ is given. Choose N such that $1/N < \varepsilon$, and put $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \cdots$. Since $\bar{d}(x_i, y_i)/i \leq 1/N$ for $i \geq N$, $D(\mathbf{x}, \mathbf{y}) \leq \max\{\bar{d}(x_1, y_1), \cdots, \frac{1}{N}\bar{d}(x_N, y_N), \frac{1}{N}\} < \varepsilon$. That is, $V \subset B_D(\mathbf{x}, \varepsilon)$, and thus $\mathcal{T}_D$ is coarser. Hence D induces the product topology on $\mathbb{R}^\omega$.

4.5 Section 21: The Metric Topology (continued)

Theorem 21.1

Let $f : X \to Y$; let X and Y be metrizable with metrics d_X and d_Y, respectively. The continuity of f is equivalent to the requirement that given $x \in X$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$.

Remark: This is the standard δ-ε notation.

Proof: ($\Rightarrow$) Let x, and $V = B_Y(f(x), \varepsilon)$ be given. Since f is continuous, $f^{-1}(V)$ is open in X, it contains some δ-ball $B_X(x, \delta)$. Then for all y such that $d_X(x, y) < \delta$, $d_Y(f(x), f(y)) < \varepsilon$.

($\Leftarrow$) Suppose V is open. For all $f(x) \in V$, there exists $B(f(x), \varepsilon) \subset V$. By the hypothesis there exists δ such that $x \in B_X(x, \delta) \subset f^{-1}(B(f(x), \varepsilon)) \subset f^{-1}(V)$. Then $f^{-1}(V)$ is open for all open set V, and thus f is continuous.

Convergent Sequences

Lemma 21.2 (The sequence lemma)

Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is metrizable.

Definition: We say the sequence $x_1, x_2, \cdots$ of the space X ***converges*** to a point $x \in X$ if for each neighborhood U of x, there is a positive integer N such that $x_n \in U$ for all $n \geq N$.

Proof: ($\Rightarrow$) Suppose $x_n \to x$, then every neighborhood U of x contains some x_i, so $x \in \bar{A}$ (Theorem 17.5).

($\Leftarrow$) Suppose $x \in \bar{A}$, for all $n \in \mathbb{Z}_+$, choose $x_n \in B(x, 1/n) \cap A$, which exists by Theorem 17.5. Then $x_n \to x$, because for all open set U containing $B(x, \varepsilon)$, choosing N such that $1/N < \varepsilon$ gives $x_n \in U$ for all $n \geq N$.

Theorem 21.3

Let $f : X \to Y$. If the function f is continuous, then for every convergence sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds of X if metrizable.

Proof: ($\Rightarrow$) Suppose V is an arbitrary neighborhood of $f(x)$. $f^{-1}(V)$ is open, so there exists N such that $x_n \in f^{-1}(V)$ if $x \geq N$. Then $f(x_n) \in V$ for all $n \geq N$, thus $f(x_n) \to f(x)$.

($\Leftarrow$) Conversely, suppose X is metrizable and A is subset of X. For all $x \in \bar{A}$, there exists a convergent sequence $x_n \to x$ where $x_n \in A$ (Lemma 21.2). Then by the hypothesis, $f(x_n) \to f(x)$, and thus $f(x) \in \overline{f(A)}$ (Lemma 21.2). It follows that $f(\bar{A}) \subset \overline{f(A)}$, so f is continuous (Theorem 18.1).

Binary Operations Between Real Functions

Lemma 21.4 The addition, subtraction, and multiplication are continuous function from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$;

and the quotient operation is a continuous function from $\mathbb{R} \times (\mathbb{R} - \{0\})$ into $\mathbb{R}$.

Theorem 21.5

If X is a topological space, and if $f, g : X \to \mathbb{R}$ are continuous function, then $f + g$, $f - g$, and fg are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous.

Uniform Convergence

Uniform Convergence Let $f_n : X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence (f_n) ***converges uniformly*** to the function $f : X \to Y$ if given $\varepsilon > 0$, there exists an integer N such that $d(f_n(x), f(x)) < \varepsilon$ for all $n > N$ and all $x \in X$.

Theorem 21.6 (Uniform limit theorem)

Let $f_n : X \to Y$ be a sequence of continuous functions form the topological space X to the metric space Y. If (f_n) converges uniformly to f, then f is continuous.

Proof: Let $x \in X$ and a neighborhood V of x be given, then V contains $V' = (f(x) - \varepsilon, f(x) + \varepsilon)$ for some ε. By the uniform continuity, there exists N such that $d(f_n(x), f(x)) < \varepsilon/3$ for $n \geq N$, and we can choose U such that $f_N(U) \subset B(f_N(x), \varepsilon/3)$ by the continuity. It follows that for all $y \in U$,

$$d(f(x), f(y)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) < 3 \cdot (\varepsilon/3) = \varepsilon,$$

so $f(y) \in V' \subset V$ for all $y \in U$, namely $f(U) \subset V$. Hence f is continuous (Theorem 18.1).

5 Connectedness and Compactness

5.1 Section 23: Connected Spaces

Connectedness Let X be a topological space. A ***separation*** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be ***connected*** if there does not exist a separation of X.

Remark: The set U, $X - U$ is a separation of X if and only if U is a nonempty subset that is both open and closed. Then the definition of connectedness is equivalent to: A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.

If X is connected, so is any space homeomorphic to X.

Lemma 23.1

If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.

Proof: ($\Leftarrow$) Suppose A, B form separation of Y, the sets A and B are both open and closed. The closure of A in Y equals $\bar{A} \cap Y = A$ because A is closed, so $\bar{A} \cap B = \varnothing$. That is, B contains no limit points of A. WLOG, the reverse holds.

($\Rightarrow$) Suppose A, B have the properties above. Since $\bar{A} \cap B = \varnothing$, $\bar{A} \cap Y = \bar{A} \cap (A \cup B) = A$, so A is closed in Y. WLOG, B is also closed. Then A, B are open since $A = Y - B$ and $B = Y - A$.

<u>**Lemma 23.2**</u> If the sets C and D form a separation of X, and if Y is connected subspace of X, then Y lies entirely within either C or D.

Sketch of Proof: Proof by contradiction. If $C' = C \cap Y$ and $D' = D \cap Y$ are nonempty, C' and D' form a separation of Y, contradicting the connectedness of Y.

Theorem 23.3

Suppose U_α are connected subspaces of X for all $\alpha \in J$, and $\bigcap U_\alpha \neq \varnothing$. Then the union $\bigcup U_\alpha$ is connected.

Proof: Proof by contradiction. Assume X is not connected, there exists a separation U, V. WLOG, assume $\{A_\alpha\}$ has a point in common in U. Then $A_\alpha \subset U$ for all α (Lemma 23.2), so V is empty, contradicting the fact that U, V is a separation.

Theorem 23.4

Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.

Proof: Proof by contradiction. Assume U, V is a separation of B. WLOG, assume $A \subset U$ (Lemma 23.2). Then $B \subset \bar{A} \subset \bar{U}$, so B does not intersect V since $\bar{U} \cap V = \varnothing$. That is, $V \cap B = \varnothing$ contradicting the fact that U, V is a separation of B.

Theorem 23.5

The image of a connected space under a continuous map is connected.

Proof: Assume $f : X \to Y$ is continuous and $f(X)$ is not connected. There exists a separation U, V of $f(X)$. The set $f^{-1}(U), f^{-1}(V)$ are nonempty disjoint sets whose union is X, and they are both open and closed in X because f is continuous. Therefore, $f^{-1}(U), f^{-1}(V)$ is a separation of X, contradicting the fact that X is connected.

Theorem 23.6

A finite cartesian product of connected spaces is connected.

Proof: For all $(x, y) \in X \times Y$, the subsets $\{x\} \times Y$ and $X \times \{y\}$ are connected because they are homeomorphic to Y and X, respectively. Let $y_0 \in Y$ be given, define $U_x = (\{x\} \times Y) \cup (X \cap \{y_0\})$, which is connected (Theorem 23.3). Note that $X \times Y = \bigcup_{x \in X} U_x$, and the intersection of $\{U_x\}$ is nonempty, then $X \times Y$ is connected (Theorem 23.3). The desired statement follows directly by induction.

5.2 Section 24: Connected Subspaces of the Real Line

Linear Continuum A simply ordered set L having more than one element is called a ***linear continuum*** if the following hold:

(1) L has the least upper bound property.

(2) If $x < y$, there exists z such that $x < z < y$.

Theorem 24.1

If L is a linear continuum in the order topology, then L is connected and so are intervals and rays in L.

Proof: For the sake of contradiction, suppose L is not connected, then there exists a separation A, B of L. Choose $a \in A$ and $b \in B$, assume $a < b$ without loss of generality. Define $A_0 = \{x \in A \,|\, x \in [a, b]\}$ and define B_0 similarly. Put $a' = \sup A_0$ and $b' = \inf B_0$. It is obvious that $a' \leq b'$. If $a' < b'$, there exists $c \in (a', b') \subset [a, b]$ so that $c \notin A_0 \cup B_0 \subset L$, contradicting that L is a linear continuum. If $a' = b'$, then $a' = b' \in A \cap B$ by the least upper bound property, contradicting A, B form a separation since $A \cap B \neq \varnothing$. Hence L is connected.

The proof for intervals and rays in L are trivial because they are indeed the linear continuum in L.

Corollary 24.2

The real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.

Theorem 24.3 (Intermediate Value Theorem)

Let $f : X \to Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Proof: We claim $r \in f(X)$. Assume this assertion does not hold, then the rays $(-\infty, r)$ and $(r, +\infty)$ in $f(X)$ form a separation of $f(X)$, contradicting the fact that $f(X)$ is connected, which holds because f is continuous and X is connected (Theorem 23.5).

Path, Path Connected Given points x and y of the space X, a ***path*** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be ***path connected*** if every pair of points of X can be joined by a path in X.

Remark: A path-connected space X is connected, but the converse does not necessarily hold. Proof: Suppose A, B is a separation of X. choose $a \in A$ and $b \in B$, there exists a path from a to b. Then the image of f is connected and intersects both A and B, contradicting to Lemma 23.2.

5.3 Section 26: Compact Spaces

Open Cover, Compactness A collection $\mathcal{A}$ of subsets of a space X is said to ***cover*** X if the union of elements of $\mathcal{A}$ is equal to X. It is called an ***open covering*** of X it its elements are open subsets of X.

A space X is said to be ***compact*** if every open covering $\mathcal{A}$ of X contains a finite subcollection that covers X.

If Y is a subspace of X, a collection $\mathcal{A}$ of subsets of X is said to ***cover Y*** if the union of its elements contains Y.

Theorem 26.1

Let Y be a subspace of X. Then Y is compact if and only if every covering Y by sets open in X contains a finite subcollection covering Y.

Proof: ($\Rightarrow$) Suppose $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is an open cover of Y in X, then $\{A_\alpha \cap Y \mid \alpha \in J\}$ is an open cover of Y. Since Y is compact, Y has a finite subcover $\{A_{\alpha_i} \cap Y\}$. Then $\{A_{\alpha_i}\}$ is a finite subcover of Y in X.

($\Leftarrow$) Let an open cover $\mathcal{A} = \{A_\alpha \cap Y\}$ of Y be given. Since $\{A_\alpha\}$ is an open cover of Y in X, it has finite subcover $\{A_{\alpha_i}\}$ of Y by hypothesis, then $\{A_{\alpha_i} \cap Y\}$ is a finite subcover of Y, so Y is compact.

Properties of Compact Sets

Theorem 26.2

Every closed subspace of a compact space is compact.

Proof: Suppose Y is a closed subset of X. Adjoining the open set $X - Y$ to the open cover $\{U_\alpha\}$ of Y in X forms an open cover of X. By compactness, there is a finite subcover of X, removing $X - Y$ (if contained in the finite subcover of X) gives a finite subcover of Y in X.

Theorem 26.3

Every compact subspaces of a Hausdorff space is closed.

Proof: Suppose Y is a compact subspace of X. Let $x \in X - Y$ be given, for all $y \in Y$, we can choose a pair of disjoint neighborhoods U_y of x and V_y of y by the Hausdorff axiom. The collection $\{V_y\}_{y \in Y}$ is an open cover of Y, so there exists a finite subcover $\{V_{y_i}\}_{i=1}^n$ of Y. Put $U_x = \bigcap_{i=1}^n U_{y_i}$, the set U_x is neighborhood of x such that $U_x \cap Y = \varnothing$ (that is, x is an interior point). Notice that $X - Y = \bigcup_{x \in X - Y} U_x$ and the latter set is open, it follows that $X - Y$ is open. Hence Y is closed.

The following lemma is proved above in Theorem 26.3:

Lemma 26.4 If Y is a compact subspace of the Hausdorff space X and x_0 is not in Y, then there exists disjoint open sets U and V of X containing x_0 and Y, respectively.

Compactness and Function

Theorem 26.5

The image of a compact space under a continuous map is compact.

Proof: Suppose $f : X \to Y$ is continuous, and $\{U_\alpha\}$ is an open cover of Y. Since the collection $\{f^{-1}(U_\alpha)\}$ is an open cover of X since f is a continuous function, then there exists an finite subcover $\{f^{-1}(U_i)\}_{i=1}^n$ of X. Then $\{U_i\}_{i=1}^n$ is a finite subcover of Y, thus Y is compact.

Theorem 26.6

Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof: Suppose X_0 is a closed subset of X. The set X_0 is compact (Theorem 26.2), then $f(X_0)$ is compact (Theorem 26.5), it follows that $f(X_0)$ is closed (Theorem 26.3). Hence f^{-1} is continuous (Theorem 18.1), and thus f is a homeomorphism.

Compactness and Product Topology

Theorem 26.8 (The Tube Lemma) Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$ of $X \times Y$, then N contains some tube $W \times Y$ above $x_0 \times Y$, where W is a neighborhood of x_0 in X.

Proof: Suppose $x_0 \times Y$ is covered by basis elements $\{U_\alpha \times V_\alpha\}$ lying in N. Since $x_0 \times Y$ is homeomorphic to Y and is thus compact, $x_0 \times Y$ can be covered by a finite subcollection $\{U_i \times V_i\}_{i=1}^n$ (we assume each $U_i \times V_i$ intersects $x_0 \times Y$, otherwise it could be removed). Define $W = U_1 \cap \cdots \cap U_n$, the set W is open and it contains x_0. It is obvious that $W \times Y$ is open and $W \times Y \subset N$.

Theorem 26.7

The product of finitely many compact spaces is compact.

Proof: Let $\mathcal{A}$ be the open covering of $X \times Y$. Given $x \in X$, the tube $x \times Y$ can be covered by some $A_1, \cdots, A_n$ in $\mathcal{A}$, and their union $N_x = \bigcup_{i=1}^n A_i$ contains $W_x \times Y$ for some open set W_x (Lemma 26.8). Since X is compact, there exists a finite subcollection $\{W_1, \cdots, W_n\}$ that covers X, so the set $N = \bigcup_{i=1}^n N_i$ covers $\bigcup_{i=1}^n (W_i \times Y) \supset X \times Y$. Note that N_i is a finite subcollection of $\mathcal{A}$ for all i, so do their finite union N. That is, N is a finite subcover of $X \times Y$, so $X \times Y$ is compact. The original statement is trivial by induction.

Remark: Indeed, the product of infinitely many compact spaces is compact, and this theorem is known as the *Tychonoff Theorem*.

Finite Intersection Property

Finite Intersection Property A collection $\mathcal{C}$ of subsets of X is said to have the ***finite intersection property*** if for every finite subcollection $\{C_1, \cdots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Theorem 26.9 Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ of all elements of $\mathcal{C}$ is nonempty.

Sketch of Proof: The compactness means that for all collection $\mathcal{A}$ of open sets, $\bigcup_{A \in \mathcal{A}} A \Rightarrow \exists$ finite $\mathcal{A}_0 \subset \mathcal{A} : \bigcup_{A \in \mathcal{A}_0} A = X$, while the other property means that for all collection $\mathcal{C}$ of closed sets, $\forall$ finite $\mathcal{C}_0 \subset \mathcal{C}$: $\bigcup_{C \in \mathcal{C}_0} C = \varnothing \Rightarrow \bigcap_{C \in \mathcal{C}} C = \varnothing$. Note that the two properties are equivalent, which is not hard to show by taking contrapositive and complements of the statements above.

5.4 Section 27: Compact Subspaces of the Real Line

Compactness on $\mathbb{R}^n$

Theorem 27.1

Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.

Sketch of Proof: Step 1: Prove the statement: if x is a point of $[a, b]$ different from b, then there is a point $y > x$ of $[a, b]$ such that the interval can be covered by at most two elements of $\mathcal{A}$.

Step 2: Let C be the set of all points $y > a$ of $[a, b]$ such that the interval $[a, y]$ can be covered by finitely many elements of $\mathcal{A}$, and let $c = \sup C$.

Step 3: Show that $c \in C$. If $c \notin C$, there is an element containing $(d, c]$ for some d. There exists $z \in (d, c)$ such that $z \in C$, otherwise d is a smaller upper bound. Since $[a, z]$ and $(d, c]$ can be covered by finitely many elements of $\mathcal{A}$, so do there union $[a, c] = [a, z] \cup (d, c]$, contradiction.

Step 4: Show that $c = b$. If $c < b$, there exists y such that $[c, y]$ can be covered by at most two elements of $\mathcal{A}$ by Step 1, then $[a, y] = [a, c] \cup [c, y]$ can be covered by finitely many elements of $\mathcal{A}$, contradiction.

Corollary 27.2 Every closed interval in $\mathbb{R}$ is compact.

Theorem 27.3

A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and is bounded in the euclidean metric d or the square metric ρ.

Remark: The statement does not necessarily hold for every metric of $\mathbb{R}^n$.

Proof: The topology on $\mathbb{R}^n$ induced by the above two metrics are the same, it suffice to consider only ρ.

($\Rightarrow$) Suppose A is compact, it is closed (Theorem 26.3). If A is not bounded, choose $\boldsymbol{x} \in A$, then the open cover $\{B_\rho(\boldsymbol{x}, n) \mid n \in \mathbb{Z}_+\}$ has no finite subcover, contradicting to the hypothesis. Therefore, A is closed and bounded.

($\Leftarrow$) Suppose A is closed and bounded. A is a subset of a cube (k-cell), which is compact, since A is bounded. Then A is compact (Theorem 26.2).

Theorem 27.4 (Extreme Value Theorem)

Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)$ for every $x \in X$.

Proof: Since f is continuous and X is compact, the image $A = f(X)$ is compact. Assume A has no largest element, the open cover $\{(-\infty, a) \mid a \in A\}$ contains no finite subcover, contradicting the hypothesis that A

is compact. A similar argument shows that A has a smallest element.

The Lebesgue Number Lemma

Distance Let (X, d) be a metric space; let A be a nonempty subset of X. For each $x \in X$, we define the ***distance form x to A*** by the equation $d(x, A) = \inf\{d(x, a) \mid a \in A\}$.

Lemma 27.5 (The Lebesgue Number Lemma)

Let $\mathcal{A}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathcal{A}$ containing it.

Remark: The number δ is called a ***Lebesgue number*** for the covering $\mathcal{A}$.

Proof: The case where $X \in \mathcal{A}$ is trivial, assume $X \notin \mathcal{A}$. Choose a finite subcover $\{A_i\}$, define $f : X \to \mathbb{R}$ by the equation $f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, X - A_i)$. Since f is continuous and $f(x) > 0$, it has a minimum value $\delta > 0$ (Theorem 27.4). For all x, notice that $\delta \leq f(x) \leq d(x, X - A_i)$ for some i, then $B(x, \delta) \subset A_i$.

Uniform Continuity

Uniform Continuity A function f from the metric space (X, d_X) to the metric space (Y, d_Y) is said to be ***uniformly continuous*** if given $\varepsilon > 0$, there is a $\delta > 0$ such that for every pair of points $x, y \in X$, $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \varepsilon$.

Theorem 27.6 (Uniform Continuity Theorem)

Let $f : X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y). Then f is uniformly continuous.

Proof: Let $\varepsilon > 0$ be given. For each $x \in X$, choose δ_x such that $d(p, x) < \delta_x \Rightarrow d(f(p), f(x)) < \varepsilon/2$. The open cover $\{B(x, \delta_x) \mid x \in X\}$ has a finite subcover $\{B(x_i, \delta_{x_i})\}$. Put $\delta = \min_i \delta_{x_i}$. For each pair of points p, q such that $d(p, q) < \delta$, choose x_i such that $p \in B(x_i, \delta_{x_i})$, clearly $d(f(x), f(p)) < \varepsilon/2$. Note that $d(q, x_i) \leq d(p, q) + d(p, x_i) < \delta + \delta_{x_i} \leq 2\delta_{x_i}$, so $d(f(x), f(q)) < \varepsilon/2$ by continuity. Therefore,

$$d(f(p), f(q)) \leq d(f(p), f(x_i)) + d(f(x_i), f(q)) < \varepsilon/2 + \varepsilon/2 \leq \varepsilon$$

so f is uniformly continuous.

Isolated Points and Uncountable Sets

If X is a space, a point x of X is said to be ***isolated point*** of X if the one-point set $\{x\}$ is open in X.

Theorem 27.7

Let X be a nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.

Sketch of Proof: Step 1: Show that given any nonempty open set U of X and any point $x \in X$, there exists a nonempty open set V in U such that $x \notin \bar{V}$ [by the Hausdorff axiom].

Step 2: Let $f : \mathbb{Z}_+ \to X$, let $x_n = f(n)$. Apply step 1 to X to choose a nonempty open set $V_1 \subset X$ such that $x_1 \notin \bar{V}_1$, and for each $n \in \mathbb{Z}_+$, choose V_n such that $V_n \subset V_{n-1}$ and $\bar{V}_n$ does not contain x_n. Consider the nested sequence $\bar{V}_1 \supset \bar{V}_2 \supset \cdots$ of nonempty closed sets. Since X is compact, there is a point $x \in \bigcap \bar{V}_n$ (Theorem 26.9). Therefore, $x \notin f(\mathbb{Z}_+)$, implying that f is not surjective.

Corollary 27.8

Every closed interval in $\mathbb{R}$ is uncountable.

6 Countability and Separation Axioms

6.1 Section 30: The Countability Axioms

First Countability Axiom A space X is said to have a ***countable basis at x*** if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$.

A space that has a countable basis at each of its points is said to satisfy the ***first countability axiom***, or to be ***first-countable***.

Theorem 30.1

Let X be a topological space.

(a) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is first-countable.

(b) Let $f : X \to Y$. If f is continuous, then for every convergence sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$; the converse holds if X is first countable.

Remarks: This theorem is generalization of Theorem 21.2 and 21.3.

Second Countability Axiom If a space X has a countable basis for its topology (namely, if the topology X is generated by a countable set), then X is said to satisfy the ***second countability axiom***, or to be ***second-countable***.

Remark: The second countability axiom implies the first, and indeed much stronger than the first. For instance,every metrizable space is first countable (see section 21) but not necessarily second countable.

Theorem 30.2

A subspace of a first countable space is first countable, and a countable product of first countable spaces is first countable. A subspace of a second countable space is second countable, and a countable product of second countable spaces is second countable.

Proof: Suppose $\mathcal{B}$ be a countable basis for $X \supset A$, then the set $\{B \cap A \,|\, B \in \mathcal{B}\}$ is a countable basis for A. Suppose B_i is a countable basis for X_i, then $\{\prod U_i \,|\, U_i \in \mathcal{B}_i\}$, where $U_i = X_i$ for all but finite many i, is a countable basis for the product space $\prod X_i$.

The proof for first countable space is similar.

Dense A subset A of a space X is said to be ***dense*** in X if $\bar{A} = X$.

Theorem 30.3

Suppose that X has a countable basis. Then

(a) Every open covering of X contains a countable subcollection covering X. (Lindelof space)

(b) There exists a countable subset of X that is dense in X. (Separable)

Remark: The properties above are generally weaker than the second countability axiom, but become equivalent when the space is metrizable.

Proof: **(a)** Suppose $\mathcal{A}$ is an open covering, define $\mathcal{A}' = \{A_n\}_{n \in \mathbb{Z}_+}$ by choosing A_n for which $B_n \subset A_n$. Notice that $\mathcal{A}'$ is obviously countable, and it is a subcover of X since $\mathcal{B}$ is a basis thus an open cover, implied by Lemma 13.1. Therefore, $\mathcal{A}'$ is a desired subcollection.

(b) Choose $D = \{x_n\}_{n \in \mathbb{Z}_+}$ from B_n for each $n \in \mathbb{Z}_+$. For all $x \in X$, every basis element containing x intersects D, so $x \in \bar{D}$. Hence D is a countable set that is dense in X.

6.2 Section 31: The Separation Axioms

Remark: Suppose X is a topological space, X is

- $\mathcal{T}_1$ (Frechet) if every finite point set is closed (see §17),
- $\mathcal{T}_2$ if X is Hausdorff
- $\mathcal{T}_3$ if X is regular Hausdorff, and
- $\mathcal{T}_4$ if X is normal Hausdorff

The definition of regular and normal spaces are given below.

Regularity Axiom ($\mathcal{T}_3$) Suppose that one point sets are closed in X. Then X is said to be ***regular*** if for each pair consisting a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.

Normalcy Axiom ($\mathcal{T}_4$) Suppose that one point sets are closed in X. Then X is said to be ***normal*** if for each pair A, B if disjoint closed sets, there exist disjoint open sets containing x and B, respectively.

Remark: The regular space is Hausdorff, and the normal space is regular (we need to include the condition that one-point sets be closed in order for this to the case).

Lemma 31.1

Let X be a topological space, and let one-point sets in X be closed.

(a) X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\bar{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$.

Proof: (a) ($\Rightarrow$) Given a neighborhood U of x. $X - U$ is closed, so there exists open sets V_1, V_2 separating $x, X - U$. Note that $\overline{V_1}$ is disjoint from $X - U$ [otherwise, if $y \in \overline{V_1} \cap (X - U)$, V_2 intersects V_1 as a neighborhood of y by Theorem 17.5, contradiction]. Then $\bar{V}_1 \subset U$.

($\Leftarrow$) Given x and a closed set U disjoint from x. Since $X - U$ is open, there exists a neighborhood V of x such that $\bar{V} \subset X - U$. Then $V, X - \bar{V}$ separates x, U.

The proof for (b) is an analogous, with replacing the point$x$ by the set A throughout.

Theorem 31.2

(a) A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.

(b) A subspace of a regular space is regular; a product of regular spaces is regular.

Remark: There is no analogous theorem for normal spaces, see next section.

Proof: **(a)** The proof is omitted, see §17.

(b) Suppose A is a subspace of X. One-point sets are closed because A is Hausdorff by (a). Let $x \in A$ and U be a closed subset in A disjoint from x. Since $U = \bar{U} \cap A$ given that U is closed in A, $x \notin \bar{U}$. By the regularity of X, x and $\bar{U}$ can be separated by the disjoint neighborhoods V_1, V_2, then x and U can be separated by the disjoint neighborhoods $V_1 \cap A$, $V_2 \cap A$.

Suppose $X = X_\alpha$. One-point sets are closed because X is Hausdorff by (a). Let $x \in X$ and U be a neighborhood of x, then there is an neighborhood $\prod U_\alpha$ such that $x \in \prod U_\alpha \subset U$. For each α, we choose $V_\alpha = U_\alpha$ if $U_\alpha = X_\alpha$, and otherwise we choose a neighborhood V_α of x_α such that $\bar{V}_\alpha \subset U_\alpha$ since X_α is regular (Lemma 31.1). Then $V = \prod V_\alpha$ is a neighborhood of x, and $\bar{V} = \prod \bar{V}_\alpha \subset \prod U_\alpha \subset U$ (Theorem 19.5), so X is regular by Lemma 31.1.

6.3 Section 32: Normal Spaces

Theorem 32.1

Every regular space with a countable basis is normal.

Remark: The stronger version is: every regular Lindelof space is normal. We formulate a proof for this statement below.

Proof of The Stronger Form: Suppose A, B are disjoint closed subset of X. For $x \in A$, by the regularity, there is a neighborhood U disjoint from a neighborhood of B. Since $\{U_a\}$ is an open covering, it has a countable subcover $\{U_n\}_{n \in Z_+}$, and the closure is disjoint from B. Choose $\{V_n\}$ for B similarly.

Define $U_n' := U_n - \bigcup_{i=1}^n \bar{V}_i$ [motivation: remove the closure of $\{V_n\}$ in U_n] and define V_i' similarly. Put $U' := \bigcup U_n'$ and $V' := \bigcup V_n'$. It it not hard to show U' and V' are open, disjoint, and contains A and B, respectively.

Theorem 32.2

Every metrizable space is normal.

Proof: Suppose A, B are disjoint closed subset. For every $a \in A$, choose ε_a such that $B(a, \varepsilon)$ is disjoint from B, and define $U := \bigcup_{a \in A} B(a, \varepsilon_a/2)$. Similarly, V for B. Note that U and V are disjoint, since $z \in B(a, \varepsilon_a/2) \cap B(b, \varepsilon_b/2)$ implies that $d(a, b) < \varepsilon_a$ or ε_b, contradiction.

Theorem 32.3

Every compact Hausdorff space if normal.

Proof: Suppose A, B are closed subset. A, B are compact (Theorem 26.2), so for all a, there exist disjoint open sets U_a, V_a containing a and B (Lemma 26.4). By compactness, A, B can be covered by a finite collection $\{U_i\}$, $\{V_i\}$, respectively. Put $U := \bigcup U_i$ and $V := \bigcap V_i$, it is obvious that U, V are disjoint open sets containing A, B, respectively.

Theorem 32.4 Every well-ordered set X is normal in the order topology.

Remark: Indeed, every order topology is normal.

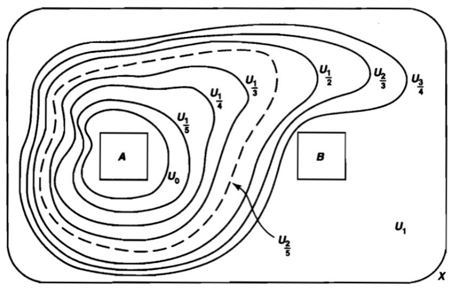
Sketch of Proof: First, we prove every interval of the form $(a, b]$ is open. Let A, B be disjoint closed sets, and denote $a_0 = \min X$. Assume $a_0 \notin A \cup B$. For all $a \in A$ such that $a \neq a_0$, there exists $(x_a, a]$ disjoint from B, and it holds for B similarly. Put $U := \bigcap_{a \in A} (x_a, a]$ and $V := \bigcap_{b \in B} (x_b, b]$, they are disjoint. Now assume $a_0 \in A \cup B$, since $\{a_0\}$ is both open and closed, append a_0 to U or V gives a pair of desired disjoint open sets.

6.4 Section 33: The Urysohn Lemma

Theorem 33.1 (Urysohn Lemma)

Let X be a normal space, let A and B be disjoint subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map $f : X \to [a, b]$ such that $f(x) = a$ for every $x \in A$, and $f(x) = b$ for every $x \in B$.

Proof: Step 1: Without loss of generality, we can let $a = 0$ and $b = 1$. Let $P = [0, 1] \cap \mathbb{Q}$. Put $U_1 = X - B$ and choose U_0 such that $A \subset U_0 \subset \overline{U_0} \subset U_1$ (Lemma 31.1). Arrange P to a sequence $\{r_1 = 1, r_2 = 0, r_3, r_4, \cdots\}$. Recursively, for all $n \geq 3$, we can choose $p, q \in \{r_1, \cdots, r_{n-1}\}$ such that $p < r_n < q$. Then we can choose U_{r_n} for which $\overline{U_p} \subset U_{r_n} \subset \overline{U_{r_n}} \subset U_q$ (Lemma 33.1). Then we obtain a collection $\{U_n\}_{n \in P}$ of neighborhoods of A such that $\overline{U_p} \subset U_q$ whenever $p < q$.



Step 2: Extend the definition of U_p to $\mathbb{Q}$ by defining $U_p = \varnothing$ if $p < 0$ and $U_p = X$ if $p > 1$. Define $f : X \to [0, 1]$ by $f(x) = \inf\{n \in \mathbb{Q} : x \in U_n\}$. It is not hard to show f is well-defined since $\mathbb{R}$ has greatest lower bound property, and $f(x) = 0$ for $x \in A$ and $f(x) = 1$ if $x \in B$.

Step 3: We need to show f is continuous. The following three lemma hold:

(a) For all $r, s \in \mathbb{Q}$ such that $0 \leq r < s$, $\overline{U_r} \subset U_s$.

The proof is obvious by cases.

(b) For all $r \geq 0$, if $x \in \overline{U_r}$, $f(x) \leq r$.

Proof: If $s > r$, $x \in U_s$ by lemma (a), so $\{n : x \in U_n\}$ contains all rational number larger than s. By the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ and the definition of infimum, $f(x) \leq r$.

(c) For all $r \geq 0$, if $x \notin U_r$, $f(x) \geq r$.

Proof: If $s < r$, $x \notin U_r$, so $\{n : x \in U_n\}$ contains no rational number less than s. Then $f(x) \leq r$.

Given a point $x_0 \in X$ and an open interval $(c, d) \subset \mathbb{R}$ containing $f(x_0)$. Since $\mathbb{Q}$ is dense, we can choose $r, s \in \mathbb{Q}$ such that $c < r < f(x_0) < s < d$. The set $V := U_s - \overline{U_r}$ is neighborhood of $f(x_0)$. By lemma (b) and (c), $r \leq f(x) \leq s$ for all $x \in V$, so $f(V) \subset (c, d)$. Hence f is continuous by Theorem 18.1.

Separated by a Continuous Function If A and B are two subsets of the topological space X, and if there is a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$, we say A and B can be ***separated by a continuous function***.

The Urysohn Lemma cannot be generalized to show that in a regular space, points can be separated from closed sets by continuous function. In step 1 of the proof when constructing U_p, we need the existence of U_p for which $U_r \subset U_p \subset \overline{U_p} \subset U_s$ if $r < p < s$, and regularity is not sufficient.

Completely Regular A space X is completely regular if one-point sets are closed in X and if for each point x_0 and each closed set A not containing x_0, there is a continuous function $f : X \to [0,1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

Remark: A normal space is completely regular (Urysohn Lemma), and a completely regular space is regular. A completely regular Hausdorff space is called Tychonoff, namely $T_{3^{1/2}}$.

Theorem 33.2

A subspace of a completely regular space is completely regular. A product of completely regular spaces is completely regular.

Proof: Let Y be a subspace of X. Suppose $x \in Y$ and A is closed in Y and disjoint from x. Note that $A = \bar{A} \cap Y$, so $x \notin \bar{A}$. Since X is completely regular, there is a continuous function $f : X \to [0,1]$ such that $f(x_0) = 0$ and $f(\bar{A}) = \{1\}$. The restriction of f to Y is a desired continuous function on Y.

Let $X = \prod X_\alpha$ be the product space, $\boldsymbol{b} \in X$, and A be a closed subset disjoint from $\boldsymbol{b}$. Choose a basis element $\prod U_\alpha$ containing $\boldsymbol{b}$; it is disjoint from A, and $U_\alpha = X_\alpha$ except for finite $\{\alpha_i\}$. For each i, choose $f_i : X_{\alpha_i} \to [0,1]$ such that $f_i(b_{\alpha_i}) = 1$ and $f_i(X - U_{\alpha_i}) = \{0\}$. Then function $f : X \to [0,1]$ defined by $f(\boldsymbol{x}) = \prod_{i=1}^n f_i(x_\alpha)$ is the desired continuous function on X, for it equals 1 at $\boldsymbol{b}$ and vanish outside $\prod U_\alpha$.

6.5 Section 34: The Urysohn Metrization Theorem

Theorem 34.1 (Urysohn Metrization Theorem)

Every regular space X with a countable basis is metrizable.

Proof: We shall prove X is metrizable by imbedding X in a metrizable space $Y = \mathbb{R}^\omega$, that is, by showing X is homeomorphic with a subspace of Y.

Step 1: We want to prove th following lemma: there exists a countable collection of continuous functions $f_n : X \to [0, 1]$ having the property that given any $x \in X$ and a neighborhood U of x, there exists an index n such that f_n is positive at x and vanish outside U.

Proof: There exists a basis element B such that $x \in B \subset \bar{B} \subset U$ by the regularity of X (Lemma 31.1). Since X is normal (Theorem 32.1), by the Urysohn Lemma, there is a continuous function $f : X \to [0, 1]$ such that $f(\bar{B}) = \{1\}$ and $f(X - U) = \{0\}$. Then f is a positive at x for all $x \in B$ and vanish outside U. Note that the basis is countable, there exists a countable collection of continuous functions $\{f_n\}_{n\in\mathbb{Z}_+}$ with the desired property.

Step 2: Let $Y = \mathbb{R}^\omega$, which is metrizable (Theorem 20.5). Define $F : X \to \mathbb{R}^\omega$ by the rule $F(x) = (f_1(x), f_2(x), \cdots)$, we assert that F is an imbedding.

F is continuous because $\mathbb{R}^\omega$ has the product topology and f_n is continuous for every n (Theorem 19.6). F is injective, since if $x \neq y$, there exists n such that $f_n(x) > 0$ and $f_n(y) = 0$. Finally, we need to show F is a homeomorphism of X onto $Z = F(X)$. F define a continuous bijection of X with Z. We need to show for each open set U in X, $F(U)$ is open in Z. Let $z_0 \in F(U)$ [want to show there exists W open in Z such that $z_0 \in W \subset F(U)$], and let x_0 be a point such that $F(x_0) = z_0$. Choose n such that $f_n(x_0) > 0$ and $f_n(X - U) = \{0\}$, and let $V = \pi_n^{-1}((0, \infty))$. It is not hard to show $z_0 \in V \cap Z$, and $W := V \cap Z \subset F(U)$ since f_n vanish outside U. Then $f(U)$ is open, it follows that F^{-1} is continuous. Hence F is a homeomorphism and thus an imbedding of X in $\mathbb{R}^\omega$.

Theorem 34.2 (Imbedding Theorem)

Let X be a space in which one-point sets are closed. Suppose that $\{f_\alpha\}_{\alpha\in J}$ is an indexed family of continuous functions $f_\alpha : x \to \mathbb{R}$ satisfying the requirement that for each point x_0 of X and each neighborhood U of x_0, there is an index α such that f_α is positive at x_0 and vanished outside U. Then the function $F : X \to \mathbb{R}^J$ defined by $F(x) = (f_\alpha(x))_{\alpha\in J}$ is an imbedding of X in $\mathbb{R}^J$. If f_α maps X into $[0, 1]$ for each α, then F imbeds X in $[0, 1]^J$.

The proof is an analogous of Step 2 of the preceding proof.

Theorem 34.3

A space X is completely regular if and only if it is homeomorphic to a subspace of $[0, 1]^J$ for some J.

7 The Fundamental Group

7.1 Section 51: Homotopy of Paths

7.1.1 Homotopy and Path of Homotopy

Homotopic, Nulhomotopic If f and f' are continuous maps of the space X into the space Y, we say that f is ***homotopic*** to f' if there is a continuous map $F : X \times I \to Y$ (where $I = [0, 1]$) such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$. The map F is called a ***homotopy*** between f and f'. If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is ***nulhomotopic***.

Remark: The homotopy $F(x, t)$ represents a continuous deforming of f to f' over time t.

For convenience we use the interval $I = [0, 1]$ as the domain for all paths. Consider the special case in which f and f' are two paths in X (note that a continuous map $f : [0, 1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$ is called a path from x_0 to x_1), there is a stronger relation between them:

Path Homotopic Two paths f and f', mapping the interval $I = [0, 1]$ into X, are said to be ***path homotopic*** if they have the same initial point x_0 and the same final point x_1, and if there is a homotopy $F : I \times I \to X$ such that $F(s, 0) = f(s)$ and $F(s, 1) = f'(s)$ for each $s, t \in I$. We call F a ***path homotopy*** between f and f'. If f is path homotopic to f', we write $f \simeq_p f'$.

Remark: The path homotopy $F(s, t)$ represents a continuous deforming of f to f' over time t, where the end points of the path remain fixed during the deformation.

Lemma 51.1

The relations $\simeq$ and $\simeq_p$ are equivalence relations.

Proof: Reflexivity and symmetry are trivial for $\simeq$ and $\simeq_p$. Suppose F and F' are the homotopy or path homotopy between f and f', and f' and f'', respectively. Defined G by the equation

$$G(x, t) = \begin{cases} F(x, 2t) & \text{for } 0 \le t \le 1/2, \\ F'(x, 2t - 1) & \text{for } 1/2 \le t \le 1. \end{cases}$$

[Motivation: We think of G as the homotopy where the first half (with respect to t) is the continuous deformation from f to f', and the second half from f' to f''.] Then G is the required homotopy between f and f'', this implies that $\simeq$ and $\simeq_p$ are transitive.

7.1.2 Product of Paths

Product of Paths If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the product $f * g$ of f and g to be the path h given by the equations

$$h(s) = \begin{cases} f(2s) & \text{for } 0 \leq s \leq 1/2, \\ f(2s-1) & \text{for } 1/2 \leq s \leq 1. \end{cases}$$

The function h is well-defined and continuous, by the pasting lemma; it is a path in X from x_0 to x_2.

Motivation: We think of h as the path whose first half is the path f and whose the second half is the path g.

The product operation on paths induces a well-defined operation on path-homotopy classes, defined by $[f] * [g] = [f * g]$. Suppose F, G are the path homotopy between f and f', between g and g', respectively; then

$$H(s,t) = \begin{cases} F(2s,t) & \text{for } 0 \leq s \leq 1/2, \\ F(2s-1,t) & \text{for } 1/2 \leq s \leq 1, \end{cases}$$

is the required path homotopy between $f * g$ and $f' * g'$.

Theorem 51.2

The operation $*$ has the following properties:

(1) (Associativity) If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$, and they are equal.

(2) (Right and left identities) Given $x \in X$ let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1, then $[f] * [e_{x_1}] = [f]$ and $[e_{x_0}] * [f] = [f]$.

(3) (Inverse) Given the path f in X from x_0 to x_1, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1-s)$. It is called the ***reverse*** of f. Then $[f] * [\bar{f}] = [e_{x_0}]$ and $[\bar{f}] * [f] = [e_{x_1}]$.

Remark: The properties above are called the ***groupiod properties*** of $*$.

Proof: Note the following two elementary facts: if f, g are two paths, $k : X \to Y$ is a continuous map, and

(a) if F is a path homotopy in X, then $k \circ F$ is a path homotopy between paths $k \circ f$ and $k \circ f'$;

(b) if $k : X \to Y$ is a continuous map and if $f(1) = g(0)$, then $k \circ (f * g) = (k \circ f) * (k \circ g)$.

(2) Let $i : I \to I$ be the identity map. Since I is convex, there is a path homotopy G between i and $e_0 * i$. Then $f \circ G$ is a path homotopy between $f \circ i = f$ and $f \circ (e_0 * i) = e_{x_0} \circ f$. The other part is an analogous.

(3) Let $\bar{i}$ denotes the reverse of i. Since I is convex, there is a homotopy between e_0 and $i * \bar{i}$. Then $f \circ H$ is a path homotopy between $f \circ e_0 = e_{x_0}$ and $f \circ (i \circ \bar{i}) = f * \bar{f}$. The other part is an analogous.

(1) Given paths f, g, h in X. Define the triple product $k_{a,b}$ (where $0 < a < b < 1$), when restricted to $[0, a]$, $[a, b]$, and $[b, 1]$, equals the positive linear maps of these intervals onto I followed by f, g, h, respectively (the *positive linear map* p of $[a, b]$ to $[c, d]$ is unique bijection of the form $p(x) = mx + k$).

Choose another pair of points c, d of I and define $k_{c,d}$. Let $p : I \to I$ be the map, when restricted to $[0, a]$, $[a, b]$, and $[b, 1]$, equals the positive linear maps onto $[0, c]$, $[c, d]$, and $[d, 1]$, respectively. It follows that $k_{c,d} \circ p = k_{a,b}$. Since I is convex, there is a path homotopy P between p and i (the identity map). Then

$k_{c,d} \circ P$ is a path homotopy in X between $k_{a,b}$ and $k_{c,d}$. Hence $f * (g * h) = k_{\frac{1}{2},\frac{3}{4}}$ and $(f * g) * h = k_{\frac{1}{4},\frac{1}{2}}$ are homotopic.

Theorem 51.3 Let f be a path in X and let $a_0, \cdots, a_n$ be numbers such that $0 = a_0 < a_1 < \cdots < a_n = 1$. Let $f_i : I \to X$ be the path that equals the positive linear map of I onto $[a_{i-1}, a_i]$ followed by f. Then $[f] = [f_1] * \cdots * [f_n]$.

7.2 Section 52: The Fundamental Group

Group Definition Review Suppose G and G' are groups:

- A ***homomorphism*** $f : G \to G'$ is a map such that $f(x \cdot y) = f(x) \cdots f(y)$ for all x, y; it satisfies the equations $f(e) = e'$ and $f(x^{-1}) = f(x)^{-1}$, where e and e' are the identities of G and G'.
- The ***kernel*** of f is the set $f^{-1}(e')$; it is a subgroup of G. The ***image*** if f is similarly a subgroup of G'.
- A homomorphism f is called a ***monomorphism*** if it is injective, called ***epimorphism*** if it is surjective, and called an ***isomorphism*** if it is bijective.

Suppose G is a group and H is a subgroup of G:

- Let xH denote the set of all products xh for $h \in H$; it is called a ***left coset*** of H in G. The collection of all such cosets forms a partition of G.
- We called H a ***normal subgroup*** if $x \cdot h \cdot x^{-1} \in H$ for each $x \in G$ and $h \in H$. In this case, we have $xH = Hx$ for each x, so out two partitions of G are the same. We denote this partition by G/H.
- If one defined $(xH) \cdot (yH) = (x \cdot y)H$, one obtains a well-defined operation on G/H that makes it a group. The group is called the ***quotient*** of G by H.

7.2.1 The Fundamental Group and Alpha-hat Function

Fundamental Group Let X be a space and x_0 be a point of X. A path in X that begins and ends at x_0 is called a loop based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the ***fundamental group*** (*first homotopy group*) of X relative to the ***base point*** x_0. It is denoted by $\pi_1(X, x_0)$.

$\hat{\alpha}$ Let α be a path in X from x_0 to x_1. We define a map $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by the equation $\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$.

Theorem 52.1

The map $\hat{\alpha}$ is a group isomorphism.

Proof: To show that $\bar{\alpha}$ is a homomorphism, we compute

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha]) = [\bar{\alpha}] * [f * g] * [\alpha] = \hat{\alpha}([f * g]).$$

To show $\bar{\alpha}$ is an isomorphism, let β denote the reverse of α, then $\hat{\beta}$ is an inverse of $\hat{\alpha}$. For each $[h] \in \pi_1(X, x_1)$, $\hat{\alpha}(\hat{\beta}([h])) = [\bar{\alpha}] * ([\alpha] * [h] * [\bar{\alpha}]) * [\alpha] = [h]$, and similarly $\hat{\beta}(\hat{\alpha}([f])) = f$ for $[f] \in \pi_1(X, x_0)$.

Corollary 52.2 If X is path connected and x_0 and x_1 are two points of X, the $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

7.2.2 Homomorphism Between Fundamental Spaces

Simply Connected A space X is said to be ***simply connected*** if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial group (group with one element) for some x_0, and hence for every $x_0 \in X$.

Lemma 52.3

In a simply connected space X, any two paths having the same initial and final points are path homotopic.

Proof: Suppose α, β are paths from x_0 to x_1. $[\alpha * \bar{\beta}]$ is a loop at x_0, so $[\alpha * \bar{\beta}] = [e_{x_0}]$ since X is simply connected. Then $[\alpha * \bar{\beta}] * [\beta] = [e_{x_0}] * [\beta] = [\beta]$, so $[\alpha] = [\beta]$.

Remark: The fundamental group is a topological invariant of the space X. If two spaces have different fundamental group, they are not homeomorphic. To prove this fact, we introduce the notion of the "homeomorphism induced by a continuous map".

Homeomorphism Induced by a Continuous Map Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by the equation $h_*([f]) = [h \circ f]$. The map h_* is called the ***homomorphism induced by h*** relative to the base point x_0.

The map h_* is well-defined, for if F is a path homotopy between f and f', then $h \circ F$ is a path homotopy between $h \circ f$ and $h \circ f'$. The fact that h_* is a homomorphism follows from $(h \circ f) * (h \circ g) = h \circ (f * g)$.

Theorem 52.4

If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.

Proof: The proof is trivial: $(k_* * h_*)([f]) = k_*([h \circ f]) = [k \circ (h \circ f)] = [(k \circ h) \circ f] = (k \circ h)_*([f])$, and similarly $i_*([f]) = [i \circ f] = [f]$.

Corollary 52.5

If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.

Proof: Let i_*, j_* denote the identity homomorphism in X and Y, respectively, then $(h^{-1})_* \circ h_* = (h^{-1} \circ h)_* = i_*$ and $h_* \circ (h^{-1})_* = (h \circ h^{-1})_* = j_*$. Therefore, $(h^{-1})_*$ is the inverse of h_*.